MAIN TOPICS

- Dynamics of Mechanical Systems,
- Degrees of Freedom and Generalized Coordinates
- Constraints
- Principle of Virtual Work
- Hamilton's Principle
- Lagrange's Equations
- Hamilton's Canonical Equations
- Gibbs-Appell's Equations
- Kane's Equations

Hamilton's Canonical Equations

MOTIVATIONS

- Obtaining first order equations directly
- Deriving equations in a simpler form
- Getting more insight in the dynamics of a system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j$$

n second order differential equations based on $q_1, q_2, \ldots, q_n$

$$p_k=\frac{\partial L}{\partial \dot{q}_k}, \quad H=\sum_{k=1}^{n} p_k \dot{q}_k - L$$

$$\Rightarrow \quad \dot{q}_k=\frac{\partial H}{\partial p_k}, \qquad \dot{p}_k=-\frac{\partial H}{\partial q_k}$$

$2n$ first econd order differential equations based on $q_1, q_2, \ldots, q_n$ and $p_1, p_2, \ldots, p_n$

GENERALIZED MOMENTUM

- **Definition:** Generalized momentum is defined as

$$p_k=\frac{\partial L}{\partial \dot{q}_k}=\frac{\partial T}{\partial \dot{q}_k}, \qquad \mathbf{p}=\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T=\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)^T$$

- Clearly $\mathbf{p}$ is a linear function of $\dot{\mathbf{q}}$

$$T=T_2+T_1+T_0=\frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}(\mathbf{q},t)\dot{\mathbf{q}}+\boldsymbol{\beta}^T(\mathbf{q},t)\dot{\mathbf{q}}+\gamma(\mathbf{q},t)$$

$$\Rightarrow \mathbf{p}=\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)^T=\mathbf{M}(\mathbf{q},t)\dot{\mathbf{q}}+\boldsymbol{\beta}(\mathbf{q},t)$$

- Since $\mathbf{M}(\mathbf{q},t)$ is always PD, it is invertible and therefore it can be concluded that $\dot{\mathbf{q}}$ is also a linear function of $\mathbf{p}$

$$\Rightarrow \dot{\mathbf{q}}=\mathbf{M}^{-1}(\mathbf{q},t)\big(\mathbf{p}-\boldsymbol{\beta}(\mathbf{q},t)\big)=\mathbf{A}(\mathbf{q},t)\mathbf{p}+\boldsymbol{\mu}(\mathbf{q},t)$$

- Since $\dot{\mathbf{q}}$ is independent clearly $\mathbf{p}$ is independent too.

Generalized Momentum

- In a non-constraint and conservative system ($\mathbf{Q}_{nc} = \mathbf{0}$)

$$\dot{p}_k = \frac{\partial L}{\partial q_k}, \qquad \dot{\mathbf{p}} = (\frac{\partial L}{\partial \mathbf{q}})^T$$

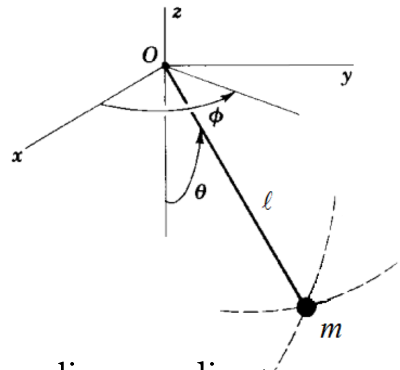
- **Definition:** q_r is called cyclic if it does not appear in the Lagrangian of the system.

- In a non-constraint and conservative with cyclic coordinates

$$\dot{p}_r = \frac{\partial L}{\partial q_r} = 0 \Rightarrow p_r = \frac{\partial L}{\partial \dot{q}_r} = \text{constant}$$

Hamilton's Canonical Equations

- Example: Spherical Pendulum



$$T = \frac{1}{2} m\left(\ell^2 \dot{\theta}^2 + \ell^2 \dot{\varphi}^2 S_\theta^2\right), \qquad V = mg\ell(1 - C_\theta)$$

$$L = \frac{1}{2} m\left(\ell^2 \dot{\theta}^2 + \ell^2 \dot{\varphi}^2 S_\theta^2\right) - mg\ell(1 - C_\theta)$$

- System is natural and conservative and φ is a cyclic coordinates, therefore immediate integral

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\ell^2 \dot{\varphi} S_\theta^2 = \text{const.} \quad \text{(angular momentum about the z-axis is constant)}$$

HAMILTON'S CANONICAL EQUATIONS

- **Definition:** Hamilton's Function is defined as

$$H = \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L \xrightarrow{\text{definition of } \mathbf{p}} H = \sum_{k=1}^{n} p_k \dot{q}_k - L = \mathbf{p}^T \dot{\mathbf{q}} - L$$

- It is easy to see that $L = T_2 + T_1 + T_0 - V \Rightarrow H = T_2 - T_0 + V$

- Using $\mathbf{p}$ instead of $\dot{\mathbf{q}}$

$$H = H(\mathbf{q}, \dot{\mathbf{q}}, t) \xrightarrow{\text{definition of } \mathbf{p}} H = H(\mathbf{q}, \mathbf{p}, t)$$

HAMILTON'S CANONICAL EQUATIONS

- Therefore $H = H(\mathbf{q}, \mathbf{p}, t) \Rightarrow \delta H = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial q_k}\delta q_k + \frac{\partial H}{\partial p_k}\delta p_k \right)$

- Also $H = \sum_{k=1}^{n} p_k \dot{q}_k - L$

$$\Rightarrow \delta H = \sum_{k=1}^{n} \left(\dot{q}_k \delta p_k + \cancel{p_k \delta \dot{q}_k} - \frac{\partial L}{\partial q_k}\delta q_k - \cancel{\frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k} \right)$$

$$= \sum_{k=1}^{n} \left(-\frac{\partial L}{\partial q_k}\delta q_k + \dot{q}_k \delta p_k \right)$$

- Comparison

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k}$$

HAMILTON'S CANONICAL EQUATIONS

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \frac{\partial H}{\partial q_k} = -\frac{\partial L}{\partial q_k}$$

- For <u>conservative and non-constrained</u> system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \;\Rightarrow\; \dot{p}_k = \frac{\partial L}{\partial q_k}$$

- Hamilton's Canonical Equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad k = 1, 2, \ldots, n$$

HAMILTON'S CANONICAL EQUATIONS

Steps:

1. Determine the system Lagrangian: $L = T - V$
2. Determine the generalized momentum: $\mathbf{p} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)^T = \left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)^T$
3. Determine $\dot{\mathbf{q}}$ based on $\mathbf{p}$ and $\mathbf{q}$: $\dot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{p} - \boldsymbol{\beta})$
4. Determine Hamiltonian as a function of $\mathbf{p}$ and $\mathbf{q}$: $H = \mathbf{p}^T \dot{\mathbf{q}} - L$
5. Derive the equations $\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k} \quad k = 1, 2, \ldots, n$

Hamilton's Canonical Equations

- Conservative and non-constrained system

$$\dot{\mathbf{q}} = (\frac{\partial H}{\partial \mathbf{p}})^T , \quad \dot{\mathbf{p}} = -(\frac{\partial H}{\partial \mathbf{q}})^T$$

- Non-constrained system

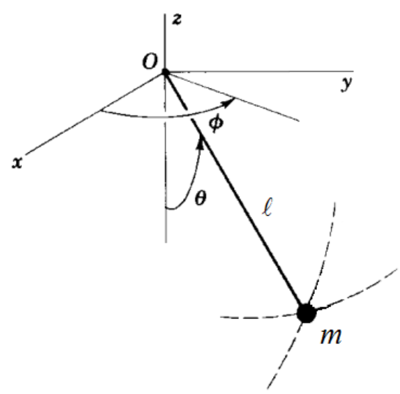
$$\dot{\mathbf{q}} = (\frac{\partial H}{\partial \mathbf{p}})^T , \quad \dot{\mathbf{p}} = -(\frac{\partial H}{\partial \mathbf{q}})^T + \mathbf{Q}_{nc}$$

- Constrained system

$$\dot{\mathbf{q}} = (\frac{\partial H}{\partial \mathbf{p}})^T , \quad \dot{\mathbf{p}} = -(\frac{\partial H}{\partial \mathbf{q}})^T - \mathbf{A}^T \boldsymbol{\lambda} + \mathbf{Q}_{nc}$$

Hamilton's Canonical Equations

- Example: Spherical Pendulum



$$T = \frac{1}{2} m \left(\ell^2 \dot{\theta}^2 + \ell^2 \dot{\varphi}^2 S_\theta^2 \right) ,$$

$$V = mg\ell(1 - C_\theta)$$

$$L = \frac{1}{2} m \left(\ell^2 \dot{\theta}^2 + \ell^2 \dot{\varphi}^2 S_\theta^2 \right) - mg\ell(1 - C_\theta)$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\ell^2 \dot{\varphi} S_\theta^2$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \dot{\theta}$$

Hamilton's Canonical Equations

- EOM

$$H = p_\varphi \dot{\varphi} + p_\theta \dot{\theta} - L \;\Rightarrow H = \frac{1}{2m\ell^2}\left(p_\theta^2 + \frac{p_\varphi^2}{S_\theta^2}\right) + mg\ell(1 - C_\theta)$$

$$\Rightarrow \begin{cases} \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} \\ \dot{\varphi} = \dfrac{\partial H}{\partial p_\varphi} \\ \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} \\ \dot{p}_\varphi = -\dfrac{\partial H}{\partial \varphi} \end{cases} \Rightarrow \begin{cases} \dot{\theta} = \dfrac{p_\theta}{m\ell^2} \\ \dot{\varphi} = \dfrac{p_\varphi}{m\ell^2 S_\theta^2} \\ \dot{p}_\theta = \dfrac{p_\varphi^2 C_\theta}{m\ell^2 S_\theta^3} - mg\ell S_\theta \\ \dot{p}_\varphi = 0 \end{cases}$$

Hamilton's Canonical Equations

For a non-constrained and conservative system

$$\frac{dH}{dt} = \sum_{k=1}^{n}\left(\cancel{\frac{\partial H}{\partial q_k}\dot{q}_k} + \cancel{\frac{\partial H}{\partial p_k}\dot{p}_k}\right) + \frac{\partial H}{\partial t} \Rightarrow \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

- If time does not appear in the system Hamiltonian or equally Lagrangian then

$$\frac{\partial H}{\partial t} = 0 \Rightarrow H = h = \text{constant} \Rightarrow \; H = T_2 - T_0 + V = \text{constant}$$

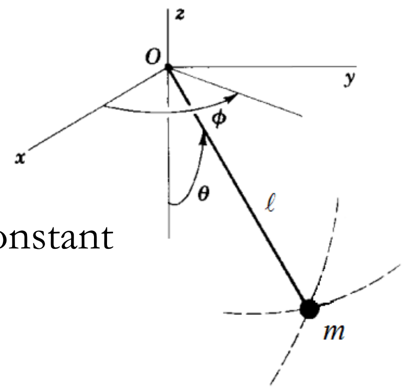
which is called Jacobi Integral

If the system is natural then $T_0 = 0$

$$H = T_2 + V = T + V = \text{constant}$$

In other word in a natural system, Hamiltonian is total energy of the system and is constant.

- Example:



$$L = \frac{1}{2} m\left(\ell^2 \dot{\theta}^2 + \ell^2 \dot{\varphi}^2 S_\theta^2\right) - mg\ell(1 - C_\theta)$$

1. System is natural therefore total energy is constant

$$H = T + V = \frac{1}{2} m\left(\ell^2 \dot{\theta}^2 + \ell^2 \dot{\varphi}^2 S_\theta^2\right) + mg\ell(1 - C_\theta) = \text{const.} = c_1$$

2. and φ cyclic coordinates therefore another immediate integral

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\ell^2 \dot{\varphi} S_\theta^2 = \text{const.}=c_2$$

(angular momentum about the z-axis is constant)

HAMILTON'S CANONICAL EQUATIONS

- Integral equations

$$m\ell^2\dot{\varphi}S_\theta^2 = c_2 \qquad\qquad \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\dot{\varphi}^2 S_\theta^2 + mg\ell(1-C_\theta) = c_1$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\dot{\varphi}S_\theta^2 = \dot{\varphi}_0 S_{\theta_0}^2 \qquad \Rightarrow \quad \dot{\theta}^2 + \dot{\varphi}^2 S_\theta^2 + \frac{2g}{\ell}(1-C_\theta) = \dot{\theta}_0^2 + \dot{\varphi}_0^2 S_{\theta_0}^2 + \frac{2g}{\ell}(1-C_{\theta_0})$$

- On other word 4 first order equations or two second order equations have been reduced to two first order equations

- Final word - difficulty in Hamilton's canonical equations: transferring from $H(\mathbf{q},\dot{\mathbf{q}},t)$ to $H(\mathbf{q},\mathbf{p},t)$ which needs calculation of $\dot{\mathbf{q}}$ based on $\mathbf{p}$

$$\dot{\mathbf{q}} = \mathbf{M}^{-1}(\mathbf{q},t)\big(\mathbf{p} - \boldsymbol{\beta}(\mathbf{q},t)\big)$$

Generalized Velocities and Generalized Speeds

MOTIVATIONS

There are two major drawback in the Lagrangian approach:

- In the case of constrained system, it does not directly give n-m independent equations of motion
- In many cases (like spatial rigid body motion) time derivatives of the generalized coordinates are not usually meaningful

Generalized speeds establish good ground to introduce other analytical approaches like Gibbs-Appell and Kane's methods that do not suffer from the above deficiencies

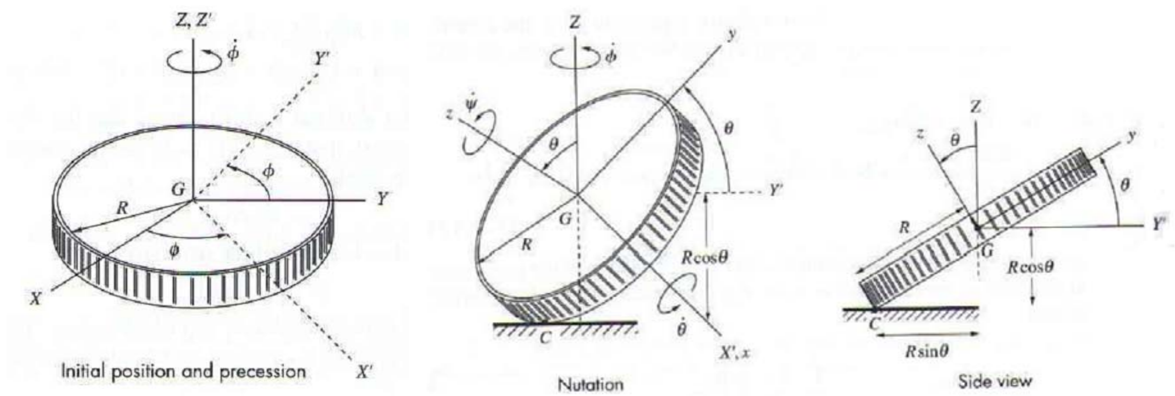
MOTIVATIONS

Example: Rolling Disk

- Rotation angles and sequences: φ, θ, ψ and 3-1-3
- Angular velocity of the disk in the body frame (x,y,z)

$$\vec{\boldsymbol{\omega}} = \dot{\theta}\hat{\mathbf{i}} + \dot{\varphi}S_\theta\hat{\mathbf{j}} + (\dot{\varphi}C_\theta + \dot{\psi})\hat{\mathbf{k}} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$$

- It is easier and meaningful to use $\omega_x, \omega_y, \omega_z$ instead of $\dot{\varphi}, \dot{\theta}, \dot{\psi}$



GENERALIZED VELOCITIES

Definition: time derivatives of $\mathbf{q}=(q_1,\ldots,q_n)^T$ defines another set of variables $\dot{\mathbf{q}}=(\dot{q}_1,\ldots,\dot{q}_n)^T$ called *generalized velocities*

- Generalized velocities at each instant defines the system velocities.

$$\vec{\mathbf{v}}_i=\vec{\mathbf{v}}_i(\mathbf{q},\dot{\mathbf{q}},t)=\sum_{k=1}^{n}\vec{\boldsymbol{\eta}}_{ik}(\mathbf{q},t)\dot{q}_k+\vec{\boldsymbol{\eta}}_{it}(\mathbf{q},t)$$

$$\vec{\boldsymbol{\omega}}_i=\vec{\boldsymbol{\omega}}_i(\mathbf{q},\dot{\mathbf{q}},t)=\sum_{k=1}^{n}\vec{\boldsymbol{\mu}}_{ik}(\mathbf{q},t)\dot{q}_k+\vec{\boldsymbol{\mu}}_{it}(\mathbf{q},t)$$

- Used to determine the generalized forces

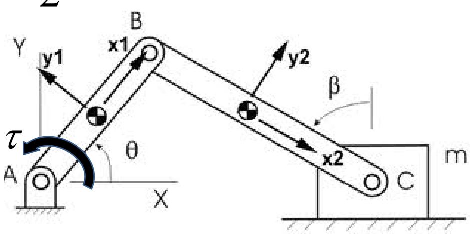
$$Q_k=\sum_{i=1}^{N}\vec{\mathbf{F}}_i\bullet\frac{\partial\vec{\mathbf{v}}_i}{\partial\dot{q}_k}+\vec{\boldsymbol{\tau}}_i\bullet\frac{\partial\vec{\boldsymbol{\omega}}_i}{\partial\dot{q}_k}\quad\Rightarrow\quad Q_k=\sum_{i=1}^{N}\vec{\mathbf{F}}_i\bullet\vec{\boldsymbol{\eta}}_{ik}+\vec{\boldsymbol{\tau}}_i\bullet\vec{\boldsymbol{\mu}}_{ik}$$

GENERALIZED VELOCITIES

- Example $q=[\theta,\beta,x]^T$

$$\text{Body 1: }\vec{\mathbf{v}}_{G_1}=\frac{\ell_1}{2}(-S_{q_1}\hat{\mathbf{i}}+C_{q_1}\hat{\mathbf{j}})\dot{q}_1\Rightarrow\vec{\boldsymbol{\eta}}_{11}=\frac{\ell_1}{2}(-S_{q_1}\hat{\mathbf{i}}+C_{q_1}\hat{\mathbf{j}})$$

$$\vec{\boldsymbol{\omega}}_1=\dot{q}_1\hat{\mathbf{k}}\Rightarrow\vec{\boldsymbol{\mu}}_{11}=\hat{\mathbf{k}}$$



$$\text{Body 2: }\vec{\mathbf{v}}_{G_2}=\frac{\ell_2}{2}(-C_{q_2}\hat{\mathbf{i}}-S_{q_2}\hat{\mathbf{j}})\dot{q}_2+\dot{q}_3\hat{\mathbf{i}}$$

$$\Rightarrow\vec{\boldsymbol{\eta}}_{22}=\frac{\ell_2}{2}(-C_{q_2}\hat{\mathbf{i}}-C_{q_2}\hat{\mathbf{j}}),\quad\vec{\boldsymbol{\eta}}_{23}=\hat{\mathbf{i}}$$

$$\vec{\boldsymbol{\omega}}_2=\dot{q}_2\hat{\mathbf{k}}\Rightarrow\vec{\boldsymbol{\mu}}_{22}=\hat{\mathbf{k}}$$

$$\text{Body 3: }\vec{\mathbf{v}}_m=\dot{q}_3\hat{\mathbf{i}}\Rightarrow\vec{\boldsymbol{\eta}}_{33}=\hat{\mathbf{i}}\qquad\vec{\boldsymbol{\omega}}_3=0$$

GENERALIZED VELOCITIES

- For a system with $m \neq 0$ constraints, the generalized velocities like the generalized coordinates are dependent through the constraint equation

$$\sum_{k=1}^{n} a_{ik}\dot{q}_k + a_{i0} = 0, \quad i = 1,\ldots,m$$

- For this system regardless of its constraints' type, one can always select $p=n-m$ independent generalized velocities

$$\dot{q}_1,\ldots,\dot{q}_p$$

GENERALIZED VELOCITIES

- Other generalized velocities can be expressed based on these independent generalized velocities

$$\sum_{k=1}^{n} a_{ik}\dot{q}_k + a_{i0} = 0, \quad i = 1,\ldots,m$$

$$\Rightarrow \dot{q}_j = \sum_{k=1}^{n-m} \alpha_{ik}\dot{q}_k + \alpha_{i0}, \quad j = n-m+1,\ldots,n$$

- Matrix presentation

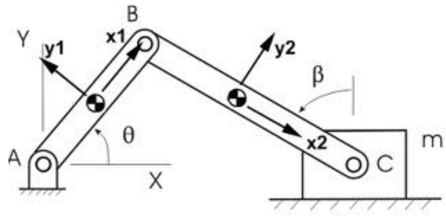
$$\mathbf{A}\dot{\mathbf{q}} + \boldsymbol{a}_0 = \mathbf{0} \xrightarrow{\text{Partitioning}} \underbrace{\begin{bmatrix}\mathbf{A}_i, & \mathbf{A}_d\end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix}\dot{\mathbf{q}}_i \\ \dot{\mathbf{q}}_d\end{bmatrix}}_{\dot{\mathbf{q}}} + \boldsymbol{a}_0 = \mathbf{0} \Rightarrow$$

$$\mathbf{A}_i\dot{\mathbf{q}}_i + \mathbf{A}_d\dot{\mathbf{q}}_d + \boldsymbol{a}_0 = \mathbf{0} \Rightarrow \dot{\mathbf{q}}_d = -\mathbf{A}_d^{-1}(\mathbf{A}_i\dot{\mathbf{q}}_i + \boldsymbol{a}_0)$$

GENERALIZED VELOCITIES

Example:

- Generalized velocities: $\dot{q}_1 = \dot{\theta},\ \dot{q}_2 = \dot{\beta},\ \dot{q}_3 = \dot{x}$
- Independent generalized speed: $\dot{q}_1 = \dot{\theta}$
- Constraint Equations:



$$\ell_1 S_\theta - \ell_2 C_\beta = 0 \quad \Rightarrow \quad \ell_1 C_{q_1}\dot{q}_1 + \ell_2 S_{q_2}\dot{q}_2 = 0$$

$$\ell_1 C_\theta + \ell_2 S_\beta = x \quad \Rightarrow \quad \ell_1 S_{q_1}\dot{q}_1 - \ell_2 C_{q_2}\dot{q}_2 + \dot{q}_3 = 0$$

$$\Rightarrow \dot{q}_2 = -\frac{\ell_1 C_{q_1}}{\ell_2 S_{q_2}}\dot{q}_1 \qquad \dot{q}_3 = -\frac{\ell_1 C_{q_1-q_2}}{S_{q_2}}\dot{q}_1$$

GENERALIZED VELOCITIES

From matrix operation

$$\mathbf{A}\dot{\mathbf{q}} = \mathbf{0} \qquad \dot{\mathbf{q}} = \begin{Bmatrix} \boxed{\dot{q}_1} \\ \boxed{\begin{matrix}\dot{q}_2 \\ \dot{q}_3\end{matrix}} \end{Bmatrix} \begin{matrix} \leftarrow \dot{\mathbf{q}}_\mathbf{i} \\ \\ \leftarrow \dot{\mathbf{q}}_\mathbf{d} \end{matrix}, \qquad \mathbf{A} = \begin{bmatrix} \underbrace{\boxed{\begin{matrix}\ell_1 C_{q_1} \\ \ell_1 S_{q_1}\end{matrix}}}_{\mathbf{A}_i} & \underbrace{\boxed{\begin{matrix}\ell_2 S_{q_2} & 0 \\ -\ell_2 C_{q_2} & 1\end{matrix}}}_{\mathbf{A}_d} \end{bmatrix}$$

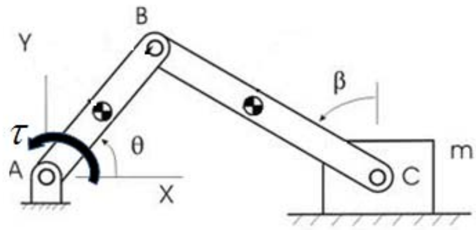
$$\Rightarrow \dot{\mathbf{q}}_\mathbf{d} = = -\mathbf{A}_d^{-1}\mathbf{A}_i\dot{\mathbf{q}}_\mathbf{i} \Rightarrow \begin{Bmatrix} \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = -\begin{Bmatrix} \dfrac{\ell_1 C_{q_1}}{\ell_2 S_{q_2}} \\ \dfrac{\ell_1 C_{q_1-q_2}}{S_{q_2}} \end{Bmatrix} \dot{q}_1$$

Generalized Velocities

- System velocities also can be expressed based on the independent generalized velocities

$$\vec{\mathbf{v}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\eta}}'_{ik}(\mathbf{q},t)\dot{q}_k + \vec{\boldsymbol{\eta}}'_{it}(\mathbf{q},t)\,, \qquad \vec{\boldsymbol{\omega}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\mu}}'_{ik}(\mathbf{q},t)\dot{q}_k + \vec{\boldsymbol{\mu}}'_{it}(\mathbf{q},t)$$

- Previous example:



$$\dot{q}_2 = -\frac{\ell_1 C_{q_1}}{\ell_2 S_{q_2}}\dot{q}_1 \qquad \dot{q}_3 = -\frac{\ell_1 C_{q_1-q_2}}{S_{q_2}}\dot{q}_1$$

Generalized Velocities

$$\text{Body 1: } \vec{\mathbf{v}}_{G_1} = \frac{\ell_1}{2}(-S_{q_1}\hat{\mathbf{i}} + C_{q_1}\hat{\mathbf{j}})\dot{q}_1 \;\Rightarrow\; \vec{\boldsymbol{\eta}}'_{11} = \frac{\ell_1}{2}(-S_{q_1}\hat{\mathbf{i}} + C_{q_1}\hat{\mathbf{j}})$$

$$\vec{\boldsymbol{\omega}}_1 = \dot{q}_1\hat{\mathbf{k}} \;\Rightarrow\; \vec{\boldsymbol{\mu}}'_{11} = \hat{\mathbf{k}}$$

$$\text{Body 2: } \vec{\mathbf{v}}_{G_2} = \frac{\ell_2}{2}(-C_{q_2}\hat{\mathbf{i}} - S_{q_2}\hat{\mathbf{j}})\dot{q}_2 + \dot{q}_3\hat{\mathbf{i}}$$

$$\Rightarrow \vec{\boldsymbol{\eta}}'_{21} = \frac{\ell_1}{2S_{q_2}}\Big((C_{q_1}C_{q_2} - 2C_{q_1-q_2})\hat{\mathbf{i}} + C_{q_1}S_{q_2}\hat{\mathbf{j}}\Big)$$

$$\vec{\boldsymbol{\omega}}_2 = \dot{q}_2\hat{\mathbf{k}} \Rightarrow \vec{\boldsymbol{\mu}}'_{21} = -\frac{\ell_1 C_{q_1}}{\ell_2 S_{q_2}}\hat{\mathbf{k}}$$

$$\text{Body 3: } \vec{\mathbf{v}}_m = \dot{q}_3\hat{\mathbf{i}} \;\Rightarrow\; \vec{\boldsymbol{\eta}}'_{33} = -\frac{\ell_1 C_{q_1-q_2}}{S_{q_2}}\hat{\mathbf{i}} \qquad \vec{\boldsymbol{\omega}}_3 = 0$$

GENERALIZED SPEEDS

Definition: instead of generalized velocities, it is possible to define n scalar variables $\mathbf{u} = (u_1, \ldots, u_n)^T$, called *generalized speeds,* in order to illustrate the system velocities

- What is the difference between generalized velocities and generalized speeds

$$\dot{\mathbf{q}} = \frac{d}{dt}\mathbf{q} \qquad\qquad \mathbf{u} \neq \frac{d}{dt}\mathbf{w}$$

- In rolling disk example:

$$\varphi, \theta, \psi \xrightarrow{\text{time differentiation}} \dot{\varphi}, \dot{\theta}, \dot{\psi}$$

$$\begin{matrix}\text{no actual variables} \\ \text{to be differentiated}\end{matrix} \Rightarrow \omega_x = \dot{\theta} \qquad \omega_y = \dot{\varphi} S_\theta \qquad \omega_z = \dot{\varphi} C_\theta + \dot{\psi}$$

GENERALIZED SPEEDS

Definition: instead of generalized velocities, it is possible to define n scalar variables $\mathbf{u} = (u_1, \ldots, u_n)^T$, called *generalized speeds,* in order to illustrate the system velocities

$$\vec{\mathbf{v}}_i = \vec{\mathbf{v}}_i(\mathbf{q}, \mathbf{u}, t) = \sum_{k=1}^{n} \vec{\boldsymbol{\gamma}}_{ik}(\mathbf{q}, t) u_k + \vec{\boldsymbol{\gamma}}_{it}(\mathbf{q}, t)$$

$$\vec{\boldsymbol{\omega}}_i = \vec{\boldsymbol{\omega}}_i(\mathbf{q}, \mathbf{u}, t) = \sum_{k=1}^{n} \vec{\boldsymbol{\beta}}_{ik}(\mathbf{q}, t) u_k + \vec{\boldsymbol{\beta}}_{it}(\mathbf{q}, t)$$

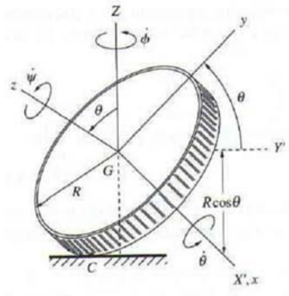
- Set of Generalized velocities is a particular set of generalized speeds

- If the system is constrained the n generalized speeds are not independent

- Minimum number of independent generalized speeds to describe the system velocities is p=n-m

GENERALIZED SPEEDS

Example: Rolling Disk with slip



- Generalized coordinates: $\mathbf{q}=(\theta,\varphi,\psi,X,Y)^T$
- System velocities:

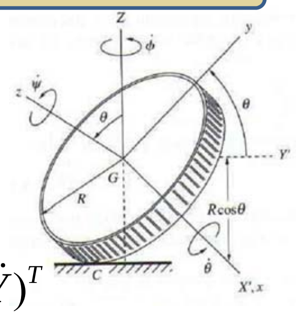
$$\vec{\boldsymbol{\omega}}=\dot{\theta}\hat{\mathbf{i}}+\dot{\varphi}S_\theta\hat{\mathbf{j}}+(\dot{\varphi}C_\theta+\dot{\psi})\hat{\mathbf{k}}$$

$$\vec{\mathbf{v}}_G=\dot{X}\hat{\mathbf{I}}+\dot{Y}\hat{\mathbf{J}}+\dot{Z}\mathbf{K}$$

$$\vec{\mathbf{v}}_C=\vec{\mathbf{v}}_G+\vec{\boldsymbol{\omega}}\times\vec{\mathbf{R}}_{C/G}=(\dot{X}-R\dot{\theta}S_\varphi S_\theta+R(\dot{\theta}C_\theta+\dot{\psi})C_\varphi)\hat{\mathbf{I}}$$

$$+(\dot{Y}+R\dot{\theta}C_\varphi S_\theta+R(\dot{\theta}C_\theta+\dot{\psi})S_\varphi)\hat{\mathbf{J}}+(\dot{Z}-R\dot{\theta}C_\theta)\hat{\mathbf{K}}$$

GENERALIZED SPEEDS



- Contact with the ground: $\dot{Z}=R\dot{\theta}C_\theta \Rightarrow Z=RS_\theta$
- Generalized speeds:

$$\mathbf{u}=(u_1=\dot{\theta},\ u_2=\dot{\varphi}S_\theta,\ u_3=\dot{\varphi}C_\theta+\dot{\psi},\ u_4=\dot{X},\ u_5=\dot{Y})^T$$

- Velocity description

$$\vec{\boldsymbol{\omega}}=\hat{\mathbf{i}}u_1+\hat{\mathbf{j}}u_2+\hat{\mathbf{k}}u_3$$

$$\vec{\mathbf{v}}_G=RC_\theta\hat{\mathbf{K}}u_1+\hat{\mathbf{I}}u_4+\hat{\mathbf{J}}u_5$$

$$\vec{\mathbf{v}}_C=RS_\theta(-S_\varphi\hat{\mathbf{I}}+C_\varphi\hat{\mathbf{J}})u_1+R(C_\varphi\hat{\mathbf{I}}+S_\varphi\hat{\mathbf{J}})u_3+\hat{\mathbf{I}}u_4+u_5\hat{\mathbf{J}}$$

GENERALIZED SPEEDS

- Note that selecting a set of generalized speeds means we can describe the system motion by $(\mathbf{q},\mathbf{u},t)$ instead of $(\mathbf{q},\dot{\mathbf{q}},t)$

- Since velocities are linear function of the generalized velocities, the transformation between generalized speeds and generalized velocities is linear, i.e.

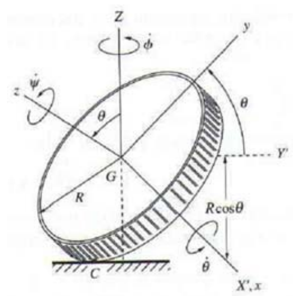
$$u_i = \sum_{j=1}^{n} \Psi_{ij}(\mathbf{q},t)\dot{q}_j + \psi_i(\mathbf{q},t) \ \xrightarrow{\text{in brief}} \ \mathbf{u} = \boldsymbol{\Psi}\dot{\mathbf{q}} + \boldsymbol{\psi}$$

$$\dot{q}_i = \sum_{j=1}^{n} \Phi_{ij}(\mathbf{q},t)u_j + \varphi_i(\mathbf{q},t) \ \xrightarrow{\text{in brief}} \ \dot{\mathbf{q}} = \boldsymbol{\Phi}\mathbf{u} + \boldsymbol{\varphi}$$

- If p=n (non-constraint) then:
  - $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are invertible and $\boldsymbol{\Psi} = \boldsymbol{\Phi}^{-1}$, $\boldsymbol{\psi} = -\boldsymbol{\Phi}^{-1}\boldsymbol{\varphi}$, $\boldsymbol{\varphi} = -\boldsymbol{\Psi}^{-1}\boldsymbol{\psi}$

GENERALIZED SPEEDS

Example: Rolling Disk with slip



- Generalized coordinates: $\mathbf{q} = (\theta,\varphi,\psi,X,Y)^T$

- Generalized speeds:

$$\mathbf{u} = (u_1 = \dot{\theta},\ u_2 = \dot{\varphi}S_\theta,\ u_3 = \dot{\varphi}C_\theta + \dot{\psi},\ u_4 = \dot{X},\ u_5 = \dot{Y})^T$$

- Transfer matrices

$$\boldsymbol{\varphi} = \boldsymbol{\psi} = \mathbf{0},\quad \boldsymbol{\Psi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & S_\theta & 0 & 0 & 0 \\ 0 & C_\theta & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},\quad \boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/S_\theta & 0 & 0 & 0 \\ 0 & -C_\theta/S_\theta & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

GENERALIZED SPEEDS

- If the system is autonomous and scleronomic then

$$\vec{\boldsymbol{\gamma}}_{it} = \vec{\boldsymbol{\beta}}_{it} = \vec{\boldsymbol{\eta}}_{it} = \vec{\boldsymbol{\mu}}_{it} = \mathbf{0}$$

$$\vec{\mathbf{v}}_i = \sum_{k=1}^{n} \vec{\boldsymbol{\eta}}_{ik} \dot{q}_k \,, \quad \vec{\mathbf{v}}_i = \sum_{k=1}^{n} \vec{\boldsymbol{\gamma}}_{ik} u_k \,,$$

$$\vec{\boldsymbol{\omega}}_i = \sum_{k=1}^{n} \vec{\boldsymbol{\mu}}_{ik} \dot{q}_k \,, \quad \vec{\boldsymbol{\omega}}_i = \sum_{k=1}^{n} \vec{\boldsymbol{\beta}}_{ik} u_k \,,$$

$$u_i = \sum_{j=1}^{n} \Psi_{ij} \dot{q}_j \quad \underset{\rightarrow}{\text{or}} \quad \mathbf{u} = \boldsymbol{\Psi} \dot{\mathbf{q}}$$

$$\dot{q}_i = \sum_{j=1}^{n} \Phi_{ij} u_j \quad \underset{\rightarrow}{\text{or}} \quad \dot{\mathbf{q}} = \boldsymbol{\Phi} \mathbf{u}$$

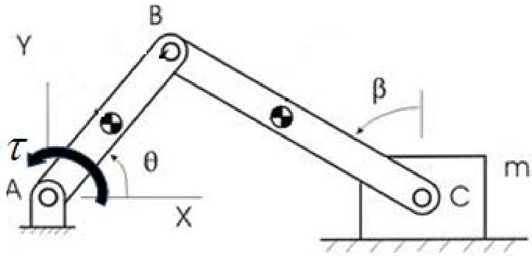
GENERALIZED SPEEDS

- Similar to generalized coordinates, set of generalized speeds is not unique
- In contrast to generalized coordinates even for non-holonomic system, we can always select $p=n-m$ independent generalized speeds.
- The set of independent generalized speeds are selected by inspection based on experience.
- Again for n-m independent generalized speeds $\mathbf{u} = (u_1, \ldots, u_{n-m})^T$

$$\dot{\mathbf{q}} = \boldsymbol{\Phi}'\mathbf{u} + \boldsymbol{\varphi}' \quad \Rightarrow \quad \vec{\mathbf{v}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\gamma}}'_{ik} u'_k + \vec{\boldsymbol{\gamma}}'_{it} \,, \quad \vec{\boldsymbol{\omega}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\beta}}'_{ik} u'_k + \vec{\boldsymbol{\beta}}'_{it}$$

GENERALIZED SPEEDS

$$\mathbf{q}=\left[\theta,\beta,\,x\right]^T \quad \mathbf{u}=\left[\dot{\theta}\right]$$



$$\begin{cases} \dot{q}_1 = u_1 \\ \dot{q}_2 = -\dfrac{\ell_1 C_{q_1}}{\ell_2 S_{q_2}} u_1 \\ \dot{q}_3 = -\dfrac{\ell_1 C_{q_1-q_2}}{S_{q_2}} u_1 \end{cases} \Rightarrow \quad \mathbf{\Phi}' = \begin{bmatrix} 1 \\ -\dfrac{\ell_1 C_{q_1}}{\ell_2 S_{q_2}} \\ -\dfrac{\ell_1 C_{q_1-q_2}}{S_{q_2}} \end{bmatrix} \qquad \boldsymbol{\varphi}' = \mathbf{0}$$

GENERALIZED SPEEDS

$$\vec{\mathbf{v}}_{G_1} = \frac{\ell_1}{2}\dot{\theta}(-S_\theta \hat{\mathbf{i}} + C_\theta \hat{\mathbf{j}}) = \underbrace{\frac{\ell_1}{2}(-S_{q_1}\hat{\mathbf{i}} + C_{q_1}\hat{\mathbf{j}})}_{\vec{\gamma}'_{11}} u_1 \qquad \vec{\mathbf{v}}_m = \dot{x}\hat{\mathbf{i}} = \underbrace{-\frac{\ell_1 C_{q_1-q_2}}{S_{q_2}}\hat{\mathbf{i}}}_{\vec{\gamma}'_{31}} u_1$$

$$\vec{\mathbf{v}}_{G_2} = \frac{\ell_2}{2}\dot{\beta}(-C_\beta \hat{\mathbf{i}} - S_\beta \hat{\mathbf{j}}) + \dot{x}\hat{\mathbf{i}} = \underbrace{\frac{\ell_1}{2S_{q_2}}\left((C_{q_1}C_{q_2} - 2C_{q_1-q_2})\hat{\mathbf{i}} + C_{q_1}S_{q_2}\hat{\mathbf{j}}\right)}_{\vec{\gamma}'_{21}} u_1$$

$$\vec{\boldsymbol{\omega}}_1 = \dot{\theta}\hat{\mathbf{k}} = \underbrace{\hat{\mathbf{k}}}_{\vec{\beta}'_{11}} u_1 \qquad \vec{\boldsymbol{\omega}}_2 = \dot{\beta}\hat{\mathbf{k}} = \underbrace{-\frac{\ell_1 C_{q_1}}{\ell_2 S_{q_2}}\hat{\mathbf{k}}}_{\vec{\beta}'_{21}} u_1 \qquad \vec{\boldsymbol{\omega}}_3 = \mathbf{0}$$

GENERALIZED SPEEDS

Consider a constraint system with n generalized coordinates and n generalized velocities, i.e.

- There are m constraints: $\mathbf{A}\dot{\mathbf{q}} + \boldsymbol{a}_0 = \mathbf{0}$
- $\dot{\mathbf{q}} = (\dot{q}_1, \ldots, \dot{q}_n)^T$ and $\mathbf{u} = (u_1, \ldots, u_n)^T$ are both dependent

$$\dot{\mathbf{q}} = \mathbf{\Phi u} + \boldsymbol{\varphi}$$

- Substituting for $\dot{\mathbf{q}}$ and $\mathbf{u}$ in constraint equations

$$\mathbf{A}\dot{\mathbf{q}} + \boldsymbol{a}_0 = \mathbf{0} \Rightarrow \mathbf{A}(\mathbf{\Phi u} + \boldsymbol{\varphi}) + \boldsymbol{a}_0 = \mathbf{0} \Rightarrow \mathbf{A}'(\mathbf{q}, t)\mathbf{u} + \boldsymbol{a}'_0(\mathbf{q}, t) = \mathbf{0}$$

$$\xrightarrow{\text{in brief}} \quad \mathbf{A}'\mathbf{u} + \boldsymbol{a}'_0 = \mathbf{0}$$

GENERALIZED SPEEDS

- A set of p independent generalized speeds can be selected

$$\mathbf{u}_{\mathrm{I}} = (u_1, \ldots, u_p)^T \qquad \mathbf{u} = \left[\mathbf{u}_{\mathrm{I}}, \mathbf{u}_{\mathrm{II}}\right]$$

- Substituting in the constraint equations

$$\mathbf{A}'\mathbf{u} + \boldsymbol{a}'_0 = \mathbf{0} \Rightarrow \mathbf{A}'_{\mathrm{I}}\mathbf{u}_{\mathrm{I}} + \mathbf{A}'_{\mathrm{II}}\mathbf{u}_{\mathrm{II}} + \boldsymbol{a}'_0 = \mathbf{0}$$

$$\Rightarrow \mathbf{u}_{\mathrm{II}} = (-\mathbf{A}'^{-1}_{\mathrm{II}}\mathbf{A}'_{\mathrm{I}})\mathbf{u}_{\mathrm{I}} + (-\mathbf{A}'^{-1}_{\mathrm{II}}\boldsymbol{a}'_0) \Rightarrow \mathbf{u}_{\mathrm{II}} = \mathbf{\Gamma}\mathbf{u}_{\mathrm{I}} + \boldsymbol{\sigma}$$

- It means $\mathbf{A}'$ should be partitioned such that $\mathbf{A}'_{\mathrm{II}}$ is nonsingular

GENERALIZED SPEEDS

- Relation between $\dot{\mathbf{q}}$ and independent set of generalized speeds

$$\mathbf{u} = \left[\mathbf{u}_{\mathrm{I}}, \mathbf{u}_{\mathrm{II}}\right] \qquad \mathbf{u}_{\mathrm{II}} = -\mathbf{A}_{\mathrm{II}}'^{-1}\mathbf{A}_{\mathrm{I}}'\mathbf{u}_{\mathrm{I}} - \mathbf{A}_{\mathrm{II}}'^{-1}\boldsymbol{a}_0'$$

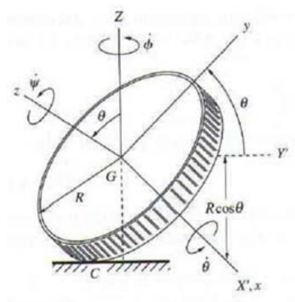
$$\dot{\mathbf{q}} = \mathbf{\Phi}\mathbf{u} + \boldsymbol{\varphi} \;\Rightarrow\; \dot{\mathbf{q}} = \mathbf{\Phi}_{\mathrm{I}}\mathbf{u}_{\mathrm{I}} + \mathbf{\Phi}_{\mathrm{II}}\mathbf{u}_{\mathrm{II}} + \boldsymbol{\varphi}$$

$$= \underbrace{\left(\mathbf{\Phi}_{\mathrm{I}} - \mathbf{\Phi}_{\mathrm{II}}\mathbf{A}_{\mathrm{II}}'^{-1}\mathbf{A}_{\mathrm{I}}'\right)}_{\mathbf{\Phi}'}\mathbf{u}_{\mathrm{I}} + \underbrace{\left(\boldsymbol{\varphi} - \mathbf{\Phi}_{\mathrm{II}}\mathbf{A}_{\mathrm{II}}'^{-1}\boldsymbol{a}_0'\right)}_{\boldsymbol{\varphi}'}$$

$$\Rightarrow \dot{\mathbf{q}} = \mathbf{\Phi}'\mathbf{u}_{\mathrm{I}} + \boldsymbol{\varphi}'$$

GENERALIZED SPEEDS

Example: Rolling Disk without slip



- Generalized coordinates: $\mathbf{q} = (\theta, \varphi, \psi, X, Y)^T$
- Generalized speeds:

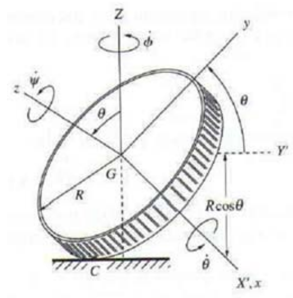
$$\mathbf{u} = (u_1 = \dot{\theta},\; u_2 = \dot{\varphi}S_\theta,\; u_3 = \dot{\varphi}C_\theta + \dot{\psi},\; u_4 = \dot{X},\; u_5 = \dot{Y})^T$$

- Constraints: rolling without slip

$$\vec{\mathbf{v}}_C = \mathbf{0} \Rightarrow \begin{cases} -RS_\varphi S_\theta u_1 + RC_\varphi u_3 + u_4 = 0 \\ RC_\varphi S_\theta u_1 + RS_\varphi u_3 + u_5 = 0 \end{cases}$$

Generalized Speeds

- Independent and dependent generalized speeds



$$\mathbf{u}_{\mathrm{I}} = (u_1, u_2, u_3)^T, \qquad \mathbf{u}_{\mathrm{II}} = (u_4, u_5)^T$$

- Constraint matrices:

$$\mathbf{A}' = \left[\begin{array}{ccc|cc} -RS_\varphi S_\theta & 0 & RC_\varphi & 1 & 0 \\ RC_\varphi S_\theta & 0 & RS_\varphi & 0 & 1 \end{array}\right] \Rightarrow \mathbf{A}'_{\mathrm{I}} = \begin{bmatrix} -RS_\varphi S_\theta & 0 & RC_\varphi \\ RC_\varphi S_\theta & 0 & RS_\varphi \end{bmatrix}, \ \mathbf{A}'_{\mathrm{II}} = \mathbf{I}_2$$

$$\boldsymbol{\Gamma} = -\mathbf{A}'^{-1}_{\mathrm{II}} \mathbf{A}'_{\mathrm{I}} = -\mathbf{A}'_{\mathrm{I}}, \quad \boldsymbol{\sigma} = \mathbf{0}$$

Generalized Speeds

- Transfer matrices $\quad \mathbf{q} = (\dot\theta, \dot\varphi, \dot\psi, \dot X, \dot Y)^T$

$$\mathbf{u} = (u_1 = \dot\theta,\ u_2 = \dot\varphi S_\theta,\ u_3 = \dot\varphi C_\theta + \dot\psi,\ u_4 = \dot X,\ u_5 = \dot Y)^T$$

$$\dot{\mathbf{q}} = \boldsymbol{\Phi}\mathbf{u} \qquad\qquad \boldsymbol{\Phi}' = \boldsymbol{\Phi}_{\mathrm{I}} - \boldsymbol{\Phi}_{\mathrm{II}} \mathbf{A}'^{-1}_{\mathrm{II}} \mathbf{A}'_{\mathrm{I}}$$

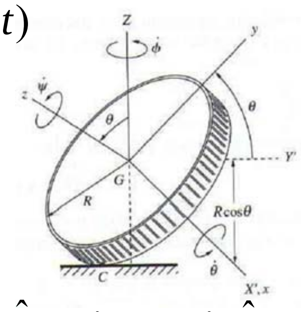
$$\boldsymbol{\Phi} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/S_\theta & 0 & 0 & 0 \\ 0 & -C_\theta/S_\theta & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \qquad \boldsymbol{\Phi}' == \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/S_\theta & 0 \\ 0 & -C_\theta/S_\theta & 1 \\ RS_\varphi S_\theta & 0 & -RC_\varphi \\ -RC_\varphi S_\theta & 0 & -RS_\varphi \end{bmatrix}$$

$$\dot{\mathbf{q}} = \boldsymbol{\Phi}'\mathbf{u}_{\mathrm{I}}$$

- Generalized speeds are a set of n scalar variables $\mathbf{u} = [u_1, \ldots, u_n]^T$, which can be used to express the system velocities

$$\vec{\mathbf{v}}_i = \vec{\mathbf{v}}_i(\mathbf{q}, \mathbf{u}, t) = \sum_{k=1}^{n} \vec{\boldsymbol{\gamma}}_{ik}(\mathbf{q}, t) u_k + \vec{\boldsymbol{\gamma}}_{it}(\mathbf{q}, t)$$

$$\vec{\boldsymbol{\omega}}_i = \vec{\boldsymbol{\omega}}_i(\mathbf{q}, \mathbf{u}, t) = \sum_{k=1}^{n} \vec{\boldsymbol{\beta}}_{ik}(\mathbf{q}, t) u_k + \vec{\boldsymbol{\beta}}_{it}(\mathbf{q}, t)$$



Example: Rolling Disk with slip

- G.C: $\mathbf{q} = (\theta, \varphi, \psi, X, Y)^T$
- S.V.: $\vec{\mathbf{v}}_G = \dot{X}\hat{\mathbf{I}} + \dot{Y}\hat{\mathbf{J}} + R\dot{\theta}C_\theta \hat{\mathbf{K}}$ $\qquad \vec{\boldsymbol{\omega}} = \dot{\theta}\hat{\mathbf{i}} + \dot{\varphi}S_\theta \hat{\mathbf{j}} + (\dot{\varphi}C_\theta + \dot{\psi})\hat{\mathbf{k}}$
- G.S.: $\mathbf{u} = (u_1 = \dot{\theta},\ u_2 = \dot{\varphi}S_\theta,\ u_3 = \dot{\varphi}C_\theta + \dot{\psi},\ u_4 = \dot{X},\ u_5 = \dot{Y})^T$

- There is a linear transformation between the set of generalized velocities $\dot{\mathbf{q}}$ and the generalized speeds $\mathbf{u}$; $\quad \dot{\mathbf{q}} = \boldsymbol{\Phi}\mathbf{u} + \boldsymbol{\varphi}$
- In the rolling disk example

$$\begin{cases} \dot{q}_1 = u_1 \\ \dot{q}_2 = u_2 / S_\theta \\ \dot{q}_3 = -(\cot\theta) u_2 + u_3 \\ \dot{q}_4 = u_4 \\ \dot{q}_5 = u_5 \end{cases} \Rightarrow \boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/S_\theta & 0 & 0 & 0 \\ 0 & -C_\theta / S_\theta & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{\varphi} = \mathbf{0}$$

Generalized Speeds (review from the previous lecture)

- System velocities:

$$\vec{\mathbf{v}}_G = \dot{X}\hat{\mathbf{I}} + \dot{Y}\hat{\mathbf{J}} + R\dot{\theta}C_\theta \mathbf{K} = RC_\theta \hat{\mathbf{K}}u_1 + \hat{\mathbf{I}}u_4 + \hat{\mathbf{J}}u_5$$

$$\vec{\boldsymbol{\omega}} = \dot{\theta}\hat{\mathbf{i}} + \dot{\varphi}S_\theta \hat{\mathbf{j}} + (\dot{\varphi}C_\theta + \dot{\psi})\hat{\mathbf{k}} = \hat{\mathbf{i}}u_1 + \hat{\mathbf{j}}u_2 + \hat{\mathbf{k}}u_3$$

$$\vec{\mathbf{v}}_C = (\dot{X} - R\dot{\theta}S_\varphi S_\theta + R(\dot{\theta}C_\theta + \dot{\psi})C_\varphi)\hat{\mathbf{I}} + (\dot{Y} + R\dot{\theta}C_\varphi S_\theta + R(\dot{\theta}C_\theta + \dot{\psi})S_\varphi)\hat{\mathbf{J}}$$

$$= RS_\theta(-S_\varphi \hat{\mathbf{I}} + C_\varphi \hat{\mathbf{J}})u_1 + R(C_\varphi \hat{\mathbf{I}} + S_\varphi \hat{\mathbf{J}})u_3 + \hat{\mathbf{I}}u_4 + u_5 \hat{\mathbf{J}}$$

Generalized Speeds (review from the previous lecture)

- In a constraint system it is possible to select n-m indepenedent generalized speeds ($\mathbf{u}_\mathrm{I}$) from the set of generalized speeds ($\mathbf{u}$)

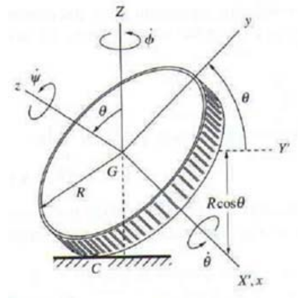
$$\mathbf{u} = [\underbrace{u_1, \ldots, u_{n-m}}_{\mathbf{u}_\mathrm{I}}, \underbrace{u_{n-m+1}, \ldots, u_n}_{\mathbf{u}_\mathrm{II}}]^T = \begin{Bmatrix} \mathbf{u}_\mathrm{I} \\ \mathbf{u}_\mathrm{II} \end{Bmatrix}$$

- Considering the constraint equations, the generalized velocities and the system velocities can be expressed only based on the independent generalized speeds (the prime coefficients)

$$\left.\begin{matrix} \mathbf{A}_\mathrm{I}\mathbf{u}_\mathrm{I} + \mathbf{A}_\mathrm{II}\mathbf{u}_\mathrm{II} + \boldsymbol{a}_0 = \mathbf{0} \\ \Downarrow \\ \mathbf{u}_\mathrm{II} = \boldsymbol{\Gamma}\mathbf{u}_\mathrm{I} + \boldsymbol{\sigma} \end{matrix}\right\} \Rightarrow \begin{cases} \dot{\mathbf{q}} = \boldsymbol{\Phi}'\mathbf{u}_\mathrm{I} + \boldsymbol{\varphi}' \\ \vec{\mathbf{v}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\gamma}}'_{ik} u'_k + \vec{\boldsymbol{\gamma}}'_{it} \Rightarrow \\ \vec{\boldsymbol{\omega}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\beta}}'_{ik} u'_k + \vec{\boldsymbol{\beta}}'_{it} \end{cases}$$

Example: Rolling Disk without slip

- G.C. : $\mathbf{q} = (\theta, \varphi, \psi, X, Y)^T$
- G.S.: $\mathbf{u} = (u_1 = \dot{\theta},\ u_2 = \dot{\varphi} S_\theta,\ u_3 = \dot{\varphi} C_\theta + \dot{\psi},\ u_4 = \dot{X},\ u_5 = \dot{Y})^T$
- C.E. (rolling without slip)



$$\vec{\mathbf{v}}_C = \mathbf{0} \Rightarrow \begin{cases} -RS_\varphi S_\theta u_1 + RC_\varphi u_3 + u_4 = 0 \\ RC_\varphi S_\theta u_1 + RS_\varphi u_3 + u_5 = 0 \end{cases}$$

- Indep. and dep. G.S.

$$\mathbf{u}_{\mathrm{I}} = (u_1, u_2, u_3)^T, \qquad \mathbf{u}_{\mathrm{II}} = (u_4, u_5)^T$$

$$\begin{cases} \dot{q}_1 = u_1 \\ \dot{q}_2 = u_2 / S_\theta \\ \dot{q}_3 = -(\cot\theta) u_2 + u_3 \\ \dot{q}_4 = RS_\varphi S_\theta u_1 - RC_\varphi u_3 \\ \dot{q}_5 = -RC_\varphi S_\theta u_1 - RS_\varphi u_3 \end{cases} \Rightarrow \mathbf{\Phi}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/S_\theta & 0 \\ 0 & -C_\theta / S_\theta & 1 \\ RS_\varphi S_\theta & 0 & -RC_\varphi \\ -RC_\varphi S_\theta & 0 & -RS_\varphi \end{bmatrix}$$

$$\vec{\mathbf{v}}_G = (RS_\varphi S_\theta \hat{\mathbf{I}} - RC_\varphi S_\theta \hat{\mathbf{J}} + RC_\theta \hat{\mathbf{K}}) u_1 + (-RC_\varphi \hat{\mathbf{I}} - RS_\varphi \hat{\mathbf{J}}) u_3$$

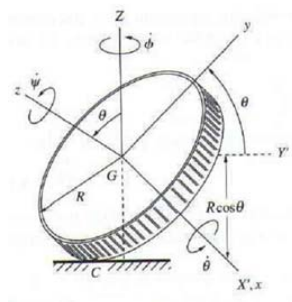
$$\vec{\boldsymbol{\omega}} = \hat{\mathbf{i}} u_1 + \hat{\mathbf{j}} u_2 + \hat{\mathbf{k}} u_3$$

GENERALIZED SPEEDS (REVIEW FROM THE PREVIOUS LECTURE)

Question: Can any 3 G.S. be selected as the I.G.S.?

- G.S.: $\mathbf{u} = (u_1 = \dot{\theta},\ u_2 = \dot{\varphi} S_\theta,\ u_3 = \dot{\varphi} C_\theta + \dot{\psi},\ u_4 = \dot{X},\ u_5 = \dot{Y})^T$

- C.E. $\vec{\mathbf{v}}_C = \mathbf{0} \Rightarrow \begin{cases} -RS_\varphi S_\theta u_1 + RC_\varphi u_3 + u_4 = 0 \\ RC_\varphi S_\theta u_1 + RS_\varphi u_3 + u_5 = 0 \end{cases}$



- Answer: **NO**

if $\mathbf{u}_{\mathrm{I}} = (u_1, u_4, u_5)^T$ and $\mathbf{u}_{\mathrm{II}} = (u_2, u_3)^T$ what is the relation between $\mathbf{u}_{\mathrm{II}}$ and $\mathbf{u}_{\mathrm{I}}$

Kane’s Equations

Definition: partial velocities

- In the Kane's method we denote and name the vector coefficients of independent generalized speeds, u_k in the velocity and angular velocity description as:

$$\vec{\mathbf{v}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\gamma}}'_{ik} u_k + \vec{\boldsymbol{\gamma}}'_{it} \xrightarrow{\text{new notation}} \vec{\mathbf{v}}_i = \sum_{k=1}^{n-m} \vec{\mathbf{v}}_i^k u_k + \vec{\mathbf{v}}_i^t$$

$$\vec{\boldsymbol{\omega}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\beta}}'_{ik} u_k + \vec{\boldsymbol{\beta}}'_{it} \xrightarrow{\text{new notation}} \vec{\boldsymbol{\omega}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\omega}}_i^k u_k + \vec{\boldsymbol{\omega}}_i^t$$

- k-th partial velocity and partial angular velocity: $\vec{\mathbf{v}}_i^k$, $\vec{\boldsymbol{\omega}}_i^k$

- time partial velocity and time partial angular velocity: $\vec{\mathbf{v}}_i^t$, $\vec{\boldsymbol{\omega}}_i^t$

- Note that

$$\vec{\mathbf{v}}_i = \vec{\mathbf{v}}_i(\mathbf{u}_\mathbf{I}, \mathbf{q}, t) = \sum_{k=1}^{n-m} \vec{\mathbf{v}}_i^k u_k + \vec{\mathbf{v}}_i^t \Rightarrow \vec{\mathbf{v}}_i^k = \frac{\partial \vec{\mathbf{v}}_i}{\partial u_k}$$

- It is comparable with

$$\dot{\vec{\mathbf{r}}}_i = (\dot{\mathbf{q}}, \mathbf{q}, t) = \sum_{k=1}^{n} \frac{\partial \vec{\mathbf{r}}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{\mathbf{r}}_i}{\partial t}$$

GENERALIZED INERTIA FORCES

- Consider a system of N particles and n-m degrees of freedom
  - For each particle:

$$m_i\,\vec{a}_i = \underbrace{\vec{\mathbf{F}}_i}_{\text{Sum of the active forces}} + \underbrace{\vec{\mathbf{F}}'_i}_{\text{Sum of constraint forces and non active forces}}$$

  - Multiply by $\vec{\mathbf{v}}_i^k$ and summing for all particles

$$\sum_{i=1}^{N} m_i\,\vec{a}_i \bullet \vec{\mathbf{v}}_i^k = \sum_{i=1}^{N} \vec{\mathbf{F}}_i \bullet \vec{\mathbf{v}}_i^k + \sum_{i=1}^{N} \vec{\mathbf{F}}'_i \bullet \vec{\mathbf{v}}_i^k\,, \quad k = 1,\ldots,n-m$$

GENERALIZED INERTIA FORCES

$$\sum_{i=1}^{N} m_i\,\vec{a}_i \bullet \vec{\mathbf{v}}_i^k = \sum_{i=1}^{N} \vec{\mathbf{F}}_i \bullet \vec{\mathbf{v}}_i^k + \sum_{i=1}^{N} \vec{\mathbf{F}}'_i \bullet \vec{\mathbf{v}}_i^k\,, \quad k = 1,\ldots,n-m$$

- Introduce **inertia force**, $\vec{\mathbf{F}}_i^* = -m_i\,\vec{a}_i$ and **generalized inertia force**

$Q_k^* = -\sum_{i=1}^{N} \vec{\mathbf{F}}_i^* \bullet \vec{\mathbf{v}}_i^k$ then

$$\underbrace{\sum_{i=1}^{N} \vec{\mathbf{F}}_i^* \bullet \vec{\mathbf{v}}_i^k}_{Q_k^*} = \underbrace{\sum_{i=1}^{N} \vec{\mathbf{F}}_i \bullet \vec{\mathbf{v}}_i^k}_{Q_k} + \underbrace{\sum_{i=1}^{N} \vec{\mathbf{F}}'_i \bullet \vec{\mathbf{v}}_i^k}_{0} \Rightarrow \quad Q_k^* = Q_k \qquad k = 1,\ldots,n-m$$

- The last sum is zero, since the virtual work of constraint forces is always zero

EQUATIONS OF MOTION

- Kane's equations of motion for a system of rigid bodies and n-m DOF corresponding to the indep. generalized speeds $\mathbf{u} = (u_1, \ldots, u_n)^T$

$$\vec{\mathbf{F}}_i^* = -m_i \vec{\boldsymbol{a}}_{ci}$$

$$\vec{\mathbf{M}}_i^* = -\frac{d}{dt}\vec{\mathbf{H}}_{ci}$$

$$Q_k^* = Q_k \quad k = 1, \ldots, n-m \xrightarrow{\text{with defintion of}} Q_k^* = -\sum_{i=1}^{N_B} \vec{\mathbf{F}}_i^* \bullet \vec{\mathbf{v}}_i^k + \vec{\mathbf{M}}_i^* \bullet \vec{\boldsymbol{\omega}}_i^k$$

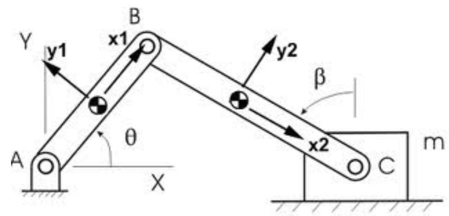
$$Q_k = \sum_{i=1}^{N_F} \vec{\mathbf{F}}_i \bullet \vec{\mathbf{v}}_i^k + \sum_{i=1}^{N_M} \vec{\mathbf{M}}_i \bullet \vec{\boldsymbol{\omega}}_i^k$$

$$\vec{\boldsymbol{a}}_{ci} = \vec{\boldsymbol{a}}_i + \ddot{\vec{\boldsymbol{\rho}}}_{ci} \qquad \frac{d}{dt}\vec{\mathbf{H}}_i = \overleftrightarrow{\mathbf{I}} \bullet \dot{\vec{\boldsymbol{\omega}}}_i + \vec{\boldsymbol{\omega}}_i \times (\overleftrightarrow{\mathbf{I}} \bullet \vec{\boldsymbol{\omega}}_i) + m_i \vec{\boldsymbol{\rho}}_{ci} \times \vec{\boldsymbol{a}}_i$$

P_i is a reference point on the body i

KANE'S EQUATIONS

Example: Slider –Crank (motion in vertical plane)



- $N_{DOF}=1$, m=2,
- G.C. $q_1 = \theta,\ q_2 = \beta,\ q_3 = x$
- G.S: $u_1 = \dot{\theta},\ u_2 = \dot{\beta},\ u_3 = \dot{x}$
- Indep. G.S. $u_1 = \dot{\theta}$
- Constraint Equations:

$$\ell_1 S_\theta - \ell_2 C_\beta = 0 \Rightarrow \ell_1 C_\theta u_1 + \ell_2 S_\beta u_2 = 0 \Rightarrow u_2 = -\frac{\ell_1 C_\theta}{\ell_2 S_\beta} u_1$$

$$\ell_1 C_\theta + \ell_2 S_\beta = x \Rightarrow \ell_1 S_\theta u_1 - \ell_2 C_\beta u_2 + \dot{x} = 0 \quad \Rightarrow u_3 = -\frac{\ell_1 C_{\theta-\beta}}{S_\beta} u_1$$

KANE'S EQUATIONS

- Velocities and Angular Velocities:

$$\vec{\mathbf{v}}_1 = \frac{\ell_1}{2}(-S_\theta\hat{\mathbf{i}} + C_\theta\hat{\mathbf{j}})u_1 \;\Rightarrow\; \vec{\mathbf{v}}_1^1 = -\frac{\ell_1}{2}S_\theta\hat{\mathbf{i}} + \frac{\ell_1}{2}C_\theta\hat{\mathbf{j}}$$

$$\vec{\mathbf{v}}_2 = \ell_1(-S_\theta\hat{\mathbf{i}} + C_\theta\hat{\mathbf{j}})u_1 + \frac{\ell_2}{2}(C_\beta\hat{\mathbf{i}} - S_\beta\hat{\mathbf{j}})u_2 \;\Rightarrow$$

$$\vec{\mathbf{v}}_2^1 = \ell_1\left(-(S_\theta + \frac{C_\theta C_\beta}{2S_\beta})\hat{\mathbf{i}} + \frac{1}{2}C_\theta\hat{\mathbf{j}}\right)$$

$$\vec{\mathbf{v}}_3 = u_3 = \frac{\ell_1 C_{\theta-\beta}}{S_\beta}u_1 \;\Rightarrow\; \vec{\mathbf{v}}_3^1 = \frac{\ell_1 C_{\theta-\beta}}{S_\beta}$$

$$\vec{\boldsymbol{\omega}}_1 = u_1\hat{\mathbf{k}} \text{ and } \vec{\boldsymbol{\omega}}_2 = u_2\hat{\mathbf{k}} \;\Rightarrow\; \vec{\boldsymbol{\omega}}_1^1 = \hat{\mathbf{k}} \text{ and } \vec{\boldsymbol{\omega}}_2^1 = -\frac{\ell_1 C_\theta}{\ell_2 S_\beta}\hat{\mathbf{k}}$$

$$\vec{\boldsymbol{\omega}}_3^1 = \vec{\mathbf{v}}_1^t = \vec{\mathbf{v}}_2^t = \vec{\mathbf{v}}_3^t = \vec{\boldsymbol{\omega}}_1^t = \vec{\boldsymbol{\omega}}_2^t = \vec{\boldsymbol{\omega}}_3^t = \mathbf{0}$$

KANE'S EQUATIONS

- Non-inertia active forces

$$\vec{\mathbf{F}}_1 = -m_1 g\frac{\ell_1}{2}\hat{\mathbf{j}}, \quad \vec{\mathbf{F}}_2 = -m_2 g\frac{\ell_2}{2}\hat{\mathbf{j}}, \quad \vec{\mathbf{F}}_3 = \mathbf{0}$$

$$\vec{\mathbf{M}}_1 = \tau\hat{\mathbf{k}}, \quad \vec{\mathbf{M}}_2 = \mathbf{0}$$

- Inertia forces: $\vec{\mathbf{F}}_i^* = -m_i\vec{a}_i, \quad \vec{\mathbf{M}}_i^* = -\left(\overleftrightarrow{\mathbf{I}}\bullet\dot{\vec{\boldsymbol{\omega}}}_i + \vec{\boldsymbol{\omega}}_i \times (\overleftrightarrow{\mathbf{I}}\bullet\vec{\boldsymbol{\omega}}_i)\right)$

$$\vec{\mathbf{F}}_1^* = -m_1\frac{\ell_1}{2}\left((-S_\theta\hat{\mathbf{i}} + C_\theta\hat{\mathbf{j}})\dot{u}_1 + (-C_\theta\hat{\mathbf{i}} - S_\theta\hat{\mathbf{j}})\dot{\theta}u_1\right)$$

$$= -m_1\frac{\ell_1}{2}\left((-S_\theta\dot{u}_1 - C_\theta u_1^2)\hat{\mathbf{i}} + (C_\theta\dot{u}_1 - S_\theta u_1^2)\hat{\mathbf{j}}\right)$$

$$\vec{\mathbf{M}}_1^* = -(I_1\hat{\mathbf{k}}\hat{\mathbf{k}}\bullet\dot{u}_1\hat{\mathbf{k}}) + \cancel{u_1\hat{\mathbf{k}}\times(I_1\hat{\mathbf{k}}\hat{\mathbf{k}}\bullet u_1\hat{\mathbf{k}})} = -I_1\dot{u}_1\hat{\mathbf{k}}$$

$$\vec{\mathbf{F}}_2^* = \frac{m_2\ell_1}{2\ell_2 S_\beta^3}\Big(\left[2S_\theta S_\beta - 2S_\theta S_\beta C_\beta^2 + C_\beta C_\theta S_\beta^2\right]\ell_2 \dot{u}_1 +$$

$$+\left[(2C_\theta S_\beta - 2C_\theta S_\beta C_\beta^2 - C_\beta S_\theta S_\beta^2)\ell_2 + \ell_1 C_\theta^2\right]u_1^2\Big)\hat{\mathbf{i}}$$

$$-\frac{3m_2\ell_1}{2}\left(C_\theta \dot{u}_1 - S_\theta u_1^2\right)\hat{\mathbf{j}}$$

$$\vec{\mathbf{M}}_2^* = \frac{m_2\ell_1}{12 S_\beta^3}\left(\ell_2 C_\theta S_\beta^2 \dot{u}_1 + \left[\ell_1 C_\theta^2 C_\beta - \ell_2 S_\theta S_\beta^2\right]u_1^2\right)\hat{\mathbf{k}}$$

$$\vec{\mathbf{F}}_3^* = \frac{m\ell_1}{\ell_2 S_\beta^3}\left(\ell_2 S_{\beta-\theta} S_\beta^2 \dot{u}_1 + \left[\ell_2 S_\beta^2 S_{\beta-\theta} + \ell_1 C_\theta^2\right]u_1^2\right)\hat{\mathbf{i}}$$

$$\vec{\mathbf{M}}_3^* = \mathbf{0}$$

- Generalized forces: $Q_1 = \sum_{i=1}^{N} \vec{\mathbf{F}}_i \bullet \vec{\mathbf{v}}_i^1 + \vec{\mathbf{M}}_i \bullet \vec{\boldsymbol{\omega}}_i^1$

Active forces and moments		Corresponding partial velocity and angular velocity
$\vec{\mathbf{F}}_1 = -m_1 g \frac{\ell_1}{2}\hat{\mathbf{j}}$	$\underrightarrow{\mathbf{v}_1}$	$\vec{\mathbf{v}}_1^1 = -\frac{\ell_1}{2} S_\theta \hat{\mathbf{i}} + \frac{\ell_1}{2} C_\theta \hat{\mathbf{j}}$
$\vec{\mathbf{F}}_2 = -m_2 g \frac{\ell_2}{2}\hat{\mathbf{j}}$	$\underrightarrow{\mathbf{v}_2}$	$\vec{\mathbf{v}}_2^1 = \ell_1\left(-(S_\theta + \frac{C_\theta C_\beta}{2S_\beta})\hat{\mathbf{i}} + \frac{1}{2} C_\theta \hat{\mathbf{j}}\right)$
$\vec{\mathbf{M}}_1 = \tau \hat{\mathbf{k}}$	$\underrightarrow{\boldsymbol{\omega}_1}$	$\vec{\boldsymbol{\omega}}_1^1 = \hat{\mathbf{k}}$
$Q_k = -\frac{g\ell_1}{4}(m_1\ell_1 + m_2\ell_2)C_\theta + \tau$		

KANE'S EQUATIONS

- Generalized inertia forces: $Q_1^* = -\sum_{i=1}^{N_B} \vec{\mathbf{F}}_i^* \bullet \vec{\mathbf{v}}_i^1 + \vec{\mathbf{M}}_i^* \bullet \vec{\boldsymbol{\omega}}_i^1$

Inertia forces and moments	Corresponding partial velocity and angular velocity
$\vec{\mathbf{F}}_1^* = -m_1 \frac{\ell_1}{2}\left((-S_\theta \dot{u}_1 - C_\theta u_1^2)\hat{\mathbf{i}} + (C_\theta \dot{u}_1 - S_\theta u_1^2)\hat{\mathbf{j}}\right)$	$\vec{\mathbf{v}}_1^1 = -\frac{\ell_1}{2} S_\theta \hat{\mathbf{i}} + \frac{\ell_1}{2} C_\theta \hat{\mathbf{j}}$
$\vec{\mathbf{F}}_2^* = \frac{m_2 \ell_1}{2\ell_2 S_\beta^3}\Big(\left[2S_\theta S_\beta - 2S_\theta S_\beta C_\beta^2 + C_\beta C_\theta S_\beta^2\right]\ell_2 \dot{u}_1 + \left[(2C_\theta S_\beta - 2C_\theta S_\beta C_\beta^2 - C_\beta S_\theta S_\beta^2)\ell_2 + \ell_1 C_\theta^2\right]u_1^2\Big)\hat{\mathbf{i}} - \frac{3m_2\ell_1}{2}\left(C_\theta \dot{u}_1 - S_\theta u_1^2\right)\hat{\mathbf{j}}$	$\vec{\mathbf{v}}_2^1 = \ell_1\left(-(S_\theta + \frac{C_\theta C_\beta}{2S_\beta})\hat{\mathbf{i}} + \frac{1}{2} C_\theta \hat{\mathbf{j}}\right)$
$\vec{\mathbf{F}}_3^* = \frac{m\ell_1}{\ell_2 S_\beta^3}\left(\ell_2 S_{\beta-\theta} S_\beta^2 \dot{u}_1 + \left[\ell_2 S_\beta^2 S_{\beta-\theta} + \ell_1 C_\theta^2\right]u_1^2\right)\hat{\mathbf{i}}$	$\vec{\mathbf{v}}_3^1 = \frac{\ell_1 C_{\theta-\beta}}{S_\beta}$

KANE'S EQUATIONS

- Generalized inertia forces: $Q_1^* = -\sum_{i=1}^{N_B} \vec{\mathbf{F}}_i^* \bullet \vec{\mathbf{v}}_i^1 + \vec{\mathbf{M}}_i^* \bullet \vec{\boldsymbol{\omega}}_i^1$

Inertia forces and moments	Corresponding partial velocity and angular velocity
$\vec{\mathbf{M}}_1^* = -I_1 \dot{u}_1 \hat{\mathbf{k}}$	$\vec{\boldsymbol{\omega}}_1^1 = \hat{\mathbf{k}}$
$\vec{\mathbf{M}}_2^* = \frac{m_2 \ell_1}{12 S_\beta^3}\left(\ell_2 C_\theta S_\beta^2 \dot{u}_1 + \left[\ell_1 C_\theta^2 C_\beta - \ell_2 S_\theta S_\beta^2\right]u_1^2\right)\hat{\mathbf{k}}$	$\vec{\boldsymbol{\omega}}_2^1 = \hat{\mathbf{k}}$

$$Q_1^* = a_1(\theta, \beta, x)\dot{u}_1 + a_2(\theta, \beta, x)u_1^2$$

KANE'S EQUATIONS

- Equations of Motion

$$a_1(\theta,\beta,x)\dot{u}_1 + a_2(\theta,\beta,x)u_1^2 + \frac{(m_1\ell_1 + m_2\ell_2)}{4} g\ell_1 C_\theta = \tau$$

$$\dot{\theta} = u_1$$

$$\dot{\beta} = -\frac{\ell_1 C_\theta}{\ell_2 S_\beta} u_1$$

$$\dot{x} = -\ell_1 C_{\beta-\theta} u_1$$

1 dynamic and 3 kinematic equations

KANE'S EQUATIONS

Procedure to derive Kane's equations

1. Define the generalized coordinates

2. Determine the system velocities according to generalized velocities

3. From the system velocities define the generalized speeds

4. From the system constraint equations define the independent generalized speeds

5. determine the linear relation between the generalized velocities and independent generalized speeds

Procedure to derive Kane's equations (cont.)

6. Express the system velocities according to ind. gen. speeds
7. Evaluate the partial velocities and angular velocities
8. Differentiate the system velocities to obtain the system acceleration
9. Substitute for time derivatives of the generalized coordinates according to independent generalized speeds
10. Determine the inertia forces and inertia moments
11. Determine the generalized forces and inertia forces

KANE'S EQUATIONS

Procedure to derive Kane's equations (cont.)

12. Derive the dynamic equations corresponding to the independent generalized speeds
13. complete the equations with the kinematic equations (linear equations between generalized velocities and independent generalized speeds) and constraint equations based on independent generalized speeds

Gibbs-Appell's Equations

GIBBS-APPELL EQUATIONS

Once independent generalized speeds are defined for a system, we can define the so called *Gibbs Function* $S(\mathbf{q},\mathbf{u}_\mathrm{I},\dot{\mathbf{u}}_\mathrm{I},t)$ as a function of generalized coordinates, independent generalized speeds and their time derivatives. It is defined by analogy to system kinetic energy

- System of Particles:

$$T=\sum_{i=1}^{N}\frac{1}{2}m_i\vec{\mathbf{v}}_i\bullet\vec{\mathbf{v}}_i \xrightarrow{\text{by analogy}} S(\mathbf{q},\mathbf{u}_\mathrm{I},\dot{\mathbf{u}}_\mathrm{I},t)=\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\vec{\mathbf{v}}}_i\bullet\dot{\vec{\mathbf{v}}}_i$$

$$\vec{\mathbf{v}}_i=\sum_{k=1}^{n-m}\vec{\boldsymbol{\gamma}}'_{ik}u_k+\vec{\boldsymbol{\gamma}}'_{it}$$

- EOM: $$\frac{\partial S}{\partial \dot{u}_k}=Q_k\,,\qquad Q_k=\sum_{i=1}^{N_F}\vec{\mathbf{F}}_i\bullet\vec{\boldsymbol{\gamma}}'_{ik}\quad k=1,\dots,n-m$$

- Kinematic Equations: $$\dot{q}_j=\sum_{k=1}^{n-m}\Phi'_{jk}(\mathbf{q},t)u_k+\varphi'_j(\mathbf{q},t),\quad j=1,\dots,n$$

GIBBS-APPELL EQUATIONS

- System of Rigid Bodies:

  - For every element dm

$$dS = \frac{1}{2} dm(\dot{\vec{\mathbf{v}}} \bullet \dot{\vec{\mathbf{v}}}) \quad \text{where} \quad \dot{\vec{\mathbf{v}}} = \dot{\vec{\mathbf{v}}}_c + \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{\rho}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\rho}})$$

  - For each body

$$S_i = \int_{\text{body}\, i} dS = \frac{1}{2} m_i \dot{\vec{\mathbf{v}}}_{ci} \bullet \dot{\vec{\mathbf{v}}}_{ci} + \left(\frac{1}{2} \overleftrightarrow{\mathbf{I}}_{ci} \bullet \dot{\vec{\boldsymbol{\omega}}}_i + \vec{\boldsymbol{\omega}}_i \times \overleftrightarrow{\mathbf{I}}_{ci} \bullet \vec{\boldsymbol{\omega}}_i \right) \bullet \dot{\vec{\boldsymbol{\omega}}}_i$$

  - For the whole system

$$S(\mathbf{q}, \mathbf{u}_{\mathrm{I}}, \dot{\mathbf{u}}_{\mathrm{I}}, t) = \sum_{i=1}^{N_B} S_i = \sum_{i=1}^{N_B} \frac{1}{2} m_i \dot{\vec{\mathbf{v}}}_{ci} \bullet \dot{\vec{\mathbf{v}}}_{ci} + \sum_{i=1}^{N_B} \left(\frac{1}{2} \overleftrightarrow{\mathbf{I}}_{ci} \bullet \dot{\vec{\boldsymbol{\omega}}}_i + \vec{\boldsymbol{\omega}}_i \times \overleftrightarrow{\mathbf{I}}_{ci} \bullet \vec{\boldsymbol{\omega}}_i \right) \bullet \dot{\vec{\boldsymbol{\omega}}}_i$$

GIBBS-APPELL EQUATIONS

- Equations of motion

$$\frac{\partial S}{\partial \dot{u}_k} = Q_k \,, \qquad Q_k = \sum_{i=1}^{N_F} \vec{\mathbf{F}}_i \bullet \vec{\boldsymbol{\gamma}}'_{ik} + \sum_{i=1}^{N_M} \vec{\mathbf{M}}_i \bullet \vec{\boldsymbol{\beta}}'_{ik} \qquad k = 1, \ldots, n-m$$

- Kinematic equations:

$$\dot{q}_j = \sum_{k=1}^{n-m} \Phi'_{jk} u_k + \varphi'_j, \quad j = 1, \ldots, n$$

Gibbs-Appell Equations

- Note: in determination of $\dot{\vec{\mathbf{v}}}_i$, $\dot{\vec{\boldsymbol{\omega}}}_i$, time derivative of $\mathbf{q}$ should be substituted by $\mathbf{u}_\mathrm{I}$

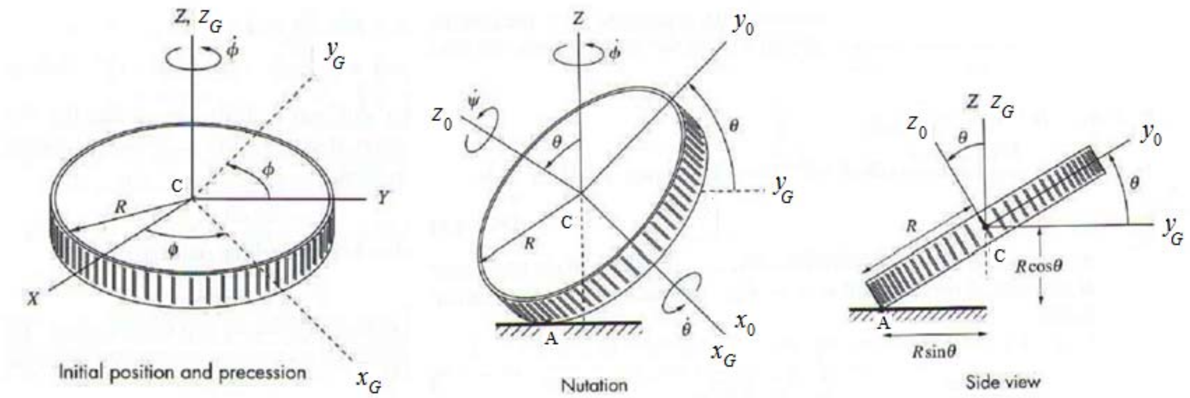
$$\vec{\mathbf{v}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\gamma}}'_{ik} u_k + \vec{\boldsymbol{\gamma}}'_{it} \Rightarrow \dot{\vec{\mathbf{v}}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\gamma}}'_{ik} \dot{u}_k + \dot{\vec{\boldsymbol{\gamma}}}'_{ik} u_k + \dot{\vec{\boldsymbol{\gamma}}}'_{it}$$

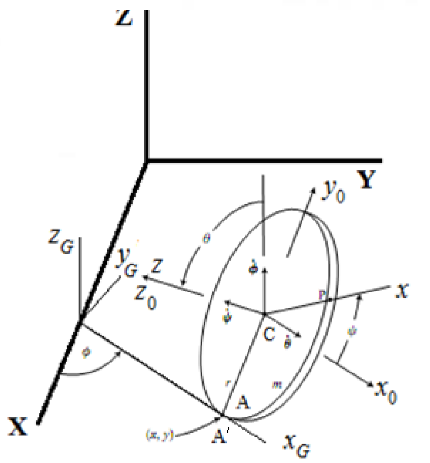
$$\vec{\boldsymbol{\omega}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\beta}}'_{ik} u_k + \vec{\boldsymbol{\beta}}'_{i0} \Rightarrow \dot{\vec{\boldsymbol{\omega}}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\beta}}'_{ik} \dot{u}_k + \dot{\vec{\boldsymbol{\beta}}}'_{ik} u_k + \dot{\vec{\boldsymbol{\beta}}}'_{it}$$

where $\dot{\vec{\boldsymbol{\gamma}}}'_{ik}, \dot{\vec{\boldsymbol{\gamma}}}'_{it}, \dot{\vec{\boldsymbol{\beta}}}'_{ik}, \dot{\vec{\boldsymbol{\beta}}}'_{it}$ are linear functions of $\mathbf{u}_\mathrm{I}$ since $\dot{\mathbf{q}} = \boldsymbol{\Phi}'\mathbf{u}_\mathrm{I} + \boldsymbol{\varphi}'$

It means S is linear in $\dot{\mathbf{u}}_\mathrm{I}$ and second order in components of $\mathbf{u}_\mathrm{I}$

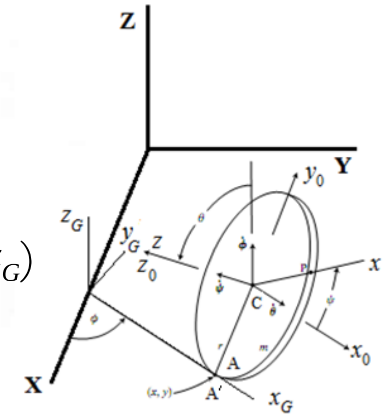
Example: Rolling Disk without Slip





EXAMPLE: ROLLING DISK WITHOUT SLIP

Frames:



- Frame I: Inertial frame (XYZ)
- Frame G: Horizontal Rotating frame $(x_G y_G z_G)$

$$\vec{\boldsymbol{\omega}}_G = \vec{\boldsymbol{\omega}}_{G/I} = \dot{\varphi}\hat{\mathbf{K}} = \dot{\varphi}\hat{\mathbf{k}}_G$$

- Frame B_0: Non-spinning body frame $(x_0 y_0 z_0)$

$$\vec{\boldsymbol{\omega}}_{B_0/G} = \dot{\theta}\hat{\mathbf{i}}_G = \dot{\theta}\hat{\mathbf{i}}_0 \Rightarrow \vec{\boldsymbol{\omega}}_{B_0} = \vec{\boldsymbol{\omega}}_{B_0/G} + \vec{\boldsymbol{\omega}}_G = \dot{\varphi}\hat{\mathbf{k}}_G + \dot{\theta}\hat{\mathbf{i}}_0$$

- Frame B: Body frame (xyz)

$$\vec{\boldsymbol{\omega}}_{B/B_0} = \dot{\psi}\hat{\mathbf{k}}_0 = \dot{\psi}\hat{\mathbf{k}} \Rightarrow \vec{\boldsymbol{\omega}}_B = \vec{\boldsymbol{\omega}}_{B/B_0} + \vec{\boldsymbol{\omega}}_{B_0} = \dot{\varphi}\hat{\mathbf{k}}_G + \dot{\theta}\hat{\mathbf{i}}_0 + \dot{\psi}\hat{\mathbf{k}}$$

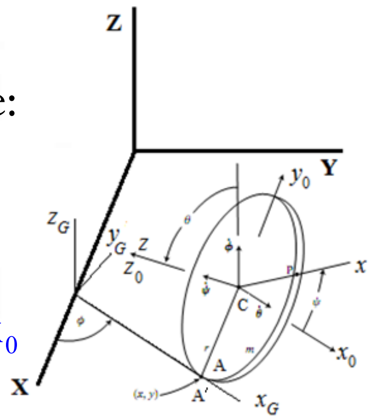
EXAMPLE: ROLLING DISK WITHOUT SLIP

Generalized coordinates: $\mathbf{q} = (\theta, \varphi, \psi, x, y)^T$

Angular velocities components in B_0 frame:

$$\vec{\boldsymbol{\omega}}_{B_0} = \dot{\varphi}\hat{\mathbf{k}}_G + \dot{\theta}\hat{\mathbf{i}}_0 = \dot{\theta}\hat{\mathbf{i}}_0 + \dot{\varphi}S_\theta\hat{\mathbf{j}}_0 + \dot{\varphi}C_\theta\hat{\mathbf{k}}_0$$

$$\vec{\boldsymbol{\omega}}_B = \dot{\varphi}\hat{\mathbf{k}}_G + \dot{\theta}\hat{\mathbf{i}}_0 + \dot{\psi}\hat{\mathbf{k}} = \dot{\theta}\hat{\mathbf{i}}_0 + \dot{\varphi}S_\theta\hat{\mathbf{j}}_0 + (\dot{\varphi}C_\theta + \dot{\psi})\hat{\mathbf{k}}_0$$



Linear velocities components in B_0 frame:

$$(x, y) = (x_{A'}, y_{A'})$$

$$\vec{\mathbf{v}}_A = \vec{\mathbf{v}}_{A'} + \vec{\mathbf{v}}_{A/A'} = \dot{x}\hat{\mathbf{I}} + \dot{y}\hat{\mathbf{J}} + R\dot{\psi}\hat{\mathbf{i}}_0 = (\dot{x} + R\dot{\psi}C_\varphi)\hat{\mathbf{I}} + (\dot{y} + R\dot{\psi}S\varphi)\hat{\mathbf{J}} = 0$$

$$\vec{\mathbf{v}}_C = \cancel{\vec{\mathbf{v}}_A} + \vec{\mathbf{v}}_{C/A} = \dot{\vec{\mathbf{r}}}_{C/A} = \frac{d}{dt}(R\hat{\mathbf{j}}_0) = \vec{\boldsymbol{\omega}}_B \times (R\hat{\mathbf{j}}_0) = -R(\dot{\varphi}C_\theta + \dot{\psi})\hat{\mathbf{i}}_0 + R\dot{\theta}\hat{\mathbf{k}}_0$$

EXAMPLE: ROLLING DISK WITHOUT SLIP

Generalized speeds: $u_1 = \dot{\theta},\ u_2 = \dot{\varphi}S_\theta,\ u_3 = \dot{\varphi}C_\theta + \dot{\psi},\ u_4 = \dot{x},\ u_5 = \dot{y}$

Kinematic relation between generalized velocities and generalized speeds

$$\dot{q}_1 = u_1 \quad \dot{q}_2 = u_2 / S_\theta \quad \dot{q}_3 = -(\cot\theta)u_2 + u_3 \quad \dot{q}_4 = u_4 \quad \dot{q}_5 = u_5$$

Constraints: $\vec{\mathbf{v}}_A = 0 \Rightarrow \dot{x} + R\dot{\psi}C_\varphi = 0,\ \ \dot{y} + R\dot{\psi}S\varphi = 0$

$$u_4 + RC_\varphi u_3 - R\cot_\theta C_\varphi u_2 = 0$$

$$u_5 + RS_\varphi u_3 - R\cot_\theta S_\varphi u_2 = 0$$

EXAMPLE: ROLLING DISK WITHOUT SLIP

Independent and dependent generalized speeds:

$$\mathbf{u}_{\mathrm{I}} = (u_1 = \dot{\theta},\ u_2 = \dot{\varphi}S_\theta,\ u_3 = \dot{\varphi}C_\theta + \dot{\psi})^T\ ,\quad \mathbf{u}_{\mathrm{II}} = (u_4 = \dot{x},\ u_5 = \dot{y})^T$$

$$u_4 = R\cot_\theta C_\varphi u_2 - RC_\varphi u_3 \qquad u_5 = R\cot_\theta S_\varphi u_2 - RS_\varphi u_3$$

Kinematic equation based on Independent generalized speeds

$$\begin{cases} \dot{q}_1 = u_1 \\ \dot{q}_2 = u_2 / S_\theta \\ \dot{q}_3 = -(\cot\theta)u_2 + u_3 \\ \dot{q}_4 = R\cot_\theta C_\varphi u_2 - RC_\varphi u_3 \\ \dot{q}_5 = R\cot_\theta S_\varphi u_2 - RS_\varphi u_3 \end{cases}$$

GIBBS-APPELL EQUATIONS

- To construct the Gibbs equations we need

$$\frac{\partial S}{\partial \dot{u}_k} = Q_k\,, \qquad Q_k = \sum_{i=1}^{N_F} \vec{\mathbf{F}}_i \bullet \vec{\boldsymbol{\gamma}}'_{ik} + \sum_{i=1}^{N_M} \vec{\mathbf{M}}_i \bullet \vec{\boldsymbol{\beta}}'_{ik} \qquad k = 1, \ldots, n-m$$

$$S = \frac{1}{2} m_i \dot{\vec{\mathbf{v}}}_C \bullet \dot{\vec{\mathbf{v}}}_C + \left(\frac{1}{2} \overleftrightarrow{\mathbf{I}}_{ci} \bullet \dot{\vec{\boldsymbol{\omega}}}_i + \vec{\boldsymbol{\omega}}_i \times (\overleftrightarrow{\mathbf{I}}_{ci} \bullet \vec{\boldsymbol{\omega}}_i) \right) \bullet \dot{\vec{\boldsymbol{\omega}}}_i$$

$$\vec{\mathbf{v}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\gamma}}'_{ik} u_k + \vec{\boldsymbol{\gamma}}'_{it}\,, \qquad \vec{\boldsymbol{\omega}}_i = \sum_{k=1}^{n-m} \vec{\boldsymbol{\beta}}'_{ik} u_k + \vec{\boldsymbol{\beta}}'_{it}$$

EXAMPLE: ROLLING DISK WITHOUT SLIP

Linear and angular velocities based on the independent generalized velocities:

$$\vec{\mathbf{v}}_C = -R(\dot{\varphi} C_\theta + \dot{\psi})\hat{\mathbf{i}}_0 + R\dot{\theta}\hat{\mathbf{k}}_0\,, \qquad \vec{\boldsymbol{\omega}}_B = \dot{\theta}\hat{\mathbf{i}}_0 + \dot{\varphi} S_\theta \hat{\mathbf{j}}_0 + (\dot{\varphi} C_\theta + \dot{\psi})\hat{\mathbf{k}}_0$$

$$\vec{\mathbf{v}}_C = -Ru_3\hat{\mathbf{i}}_0 + Ru_1\hat{\mathbf{k}}_0 \quad \xRightarrow{\vec{\mathbf{v}}_G = \sum_{k=1}^{3} \vec{\boldsymbol{\gamma}}'_k u_k} \quad \vec{\boldsymbol{\gamma}}'_1 = R\hat{\mathbf{k}}_0\,,\ \vec{\boldsymbol{\gamma}}'_3 = -R\cot_\theta \hat{\mathbf{i}}_0\,,\ \vec{\boldsymbol{\gamma}}'_2 = \mathbf{0},\ \vec{\boldsymbol{\gamma}}'_t = \mathbf{0}$$

$$\vec{\boldsymbol{\omega}}_B = u_1\hat{\mathbf{i}}_0 + u_2\hat{\mathbf{j}}_0 + u_3\hat{\mathbf{k}}_0 \quad \xRightarrow{\vec{\boldsymbol{\omega}}_B = \sum_{k=1}^{3} \vec{\boldsymbol{\beta}}'_k u_k} \quad \vec{\boldsymbol{\beta}}'_1 = \hat{\mathbf{i}}_0\,,\ \vec{\boldsymbol{\beta}}'_2 = \hat{\mathbf{j}}_0\,,\ \vec{\boldsymbol{\beta}}'_3 = \hat{\mathbf{k}}_0\,,\ \vec{\boldsymbol{\beta}}'_t = \mathbf{0}$$

EXAMPLE: ROLLING DISK WITHOUT SLIP

Gibbs function: $S=\sum_{i=1}^{N}\frac{1}{2}m\dot{\vec{\mathbf{v}}}_C\bullet\dot{\vec{\mathbf{v}}}_C+\sum_{i=1}^{N}\left(\frac{1}{2}\dot{\vec{\boldsymbol{\omega}}}_B\bullet\dot{\vec{\mathbf{H}}}_C+(\vec{\boldsymbol{\omega}}_B\times\vec{\mathbf{H}}_C)\bullet\dot{\vec{\boldsymbol{\omega}}}_B\right)$

$$\vec{\mathbf{v}}_C=-Ru_3\hat{\mathbf{i}}_0+Ru_1\hat{\mathbf{k}}_0\Rightarrow\dot{\vec{\mathbf{v}}}_C=\frac{{}^I d}{dt}\vec{\mathbf{v}}_C=\frac{{}^{B_0}d}{dt}\vec{\mathbf{v}}_C+\vec{\boldsymbol{\omega}}_{B_0}\times\vec{\mathbf{v}}_C$$

$$\dot{\vec{\mathbf{v}}}_C=(-R\dot{u}_3+Ru_1u_2)\hat{\mathbf{i}}_0-(-Ru_2u_3\cot_\theta+Ru_1^2)\hat{\mathbf{j}}_0+(R\dot{u}_1+Ru_2u_3)\hat{\mathbf{k}}_0$$

$$\vec{\boldsymbol{\omega}}_B=u_1\hat{\mathbf{i}}_0+u_2\hat{\mathbf{j}}_0+u_3\hat{\mathbf{k}}_0\Rightarrow\dot{\vec{\boldsymbol{\omega}}}_B=\frac{{}^I d}{dt}\vec{\boldsymbol{\omega}}_B=\frac{{}^{B_0}d}{dt}\vec{\boldsymbol{\omega}}_B+\vec{\boldsymbol{\omega}}_{B_0}\times\vec{\boldsymbol{\omega}}_B$$

$$\dot{\vec{\boldsymbol{\omega}}}_B=(\dot{u}_1-u_2^2\cot_\theta+u_2u_3)\hat{\mathbf{i}}_0-(\dot{u}_2+u_1u_2\cot_\theta-u_1u_3)\hat{\mathbf{j}}_0+\dot{u}_3\hat{\mathbf{k}}_0$$

EXAMPLE: ROLLING DISK WITHOUT SLIP

Gibbs function: $S=\sum_{i=1}^{N}\frac{1}{2}m\dot{\vec{\mathbf{v}}}_C\bullet\dot{\vec{\mathbf{v}}}_C+\sum_{i=1}^{N}\left(\frac{1}{2}\dot{\vec{\boldsymbol{\omega}}}_B\bullet\dot{\vec{\mathbf{H}}}_C+(\vec{\boldsymbol{\omega}}_B\times\vec{\mathbf{H}}_C)\bullet\dot{\vec{\boldsymbol{\omega}}}_B\right)$

$$\vec{\mathbf{H}}_C=\vec{\mathbf{I}}_C\bullet\vec{\boldsymbol{\omega}}_B=(I_x\hat{\mathbf{i}}_0\hat{\mathbf{i}}_0+I_y\hat{\mathbf{j}}_0\hat{\mathbf{j}}_0+I_z\hat{\mathbf{k}}_0\hat{\mathbf{k}}_0)\bullet(u_1\hat{\mathbf{i}}_0+u_2\hat{\mathbf{j}}_0+u_3\hat{\mathbf{k}}_0)$$

$$=I_xu_1\hat{\mathbf{i}}_0+I_yu_2\hat{\mathbf{j}}_0+I_zu_3\hat{\mathbf{k}}_0\ ,\qquad I_x=I_y=I_t=I_0\qquad I_z=I_a=2I_0$$

$$\vec{\boldsymbol{\omega}}_B\times\vec{\mathbf{H}}_C=I_0(u_2u_3\hat{\mathbf{i}}_0-u_1u_3\hat{\mathbf{j}}_0)$$

$$\dot{\vec{\mathbf{H}}}_C=\frac{{}^I d}{dt}\vec{\mathbf{H}}_C=\frac{{}^{B_0}d}{dt}\vec{\mathbf{H}}_C+\vec{\boldsymbol{\omega}}_{B_0}\times\vec{\mathbf{H}}_C$$

$$=I_0\left((\dot{u}_1-u_2^2\cot_\theta+2u_2u_3)\hat{\mathbf{i}}_0+(\dot{u}_2+u_1u_2\cot_\theta-2u_1u_3)\hat{\mathbf{j}}_0+2\dot{u}_3\hat{\mathbf{k}}_0\right)$$

EXAMPLE: ROLLING DISK WITHOUT SLIP

Gibbs function: $S=\sum_{i=1}^{N}\frac{1}{2}m\dot{\vec{\mathbf{v}}}_C\cdot\dot{\vec{\mathbf{v}}}_C+\sum_{i=1}^{N}\left(\frac{1}{2}\dot{\vec{\boldsymbol{\omega}}}_B\cdot\dot{\vec{\mathbf{H}}}_C+(\vec{\boldsymbol{\omega}}_B\times\vec{\mathbf{H}}_C)\cdot\dot{\vec{\boldsymbol{\omega}}}_B\right)$

$$S=I_0u_3^2+\frac{1}{2}I_0(u_2u_3\dot{u}_1-u_1u_3\dot{u}_2)+2I_0\left((\dot{u}_3-u_1u_2)^2+(\dot{u}_1+u_2u_3)^2\right)$$

$$+\frac{1}{2}I_0\Big((\dot{u}_1-u_2^2\cot_\theta+2u_2u_3)(\dot{u}_1-u_2^2\cot_\theta+u_2u_3)$$

$$+(\dot{u}_2+u_1u_2\cot_\theta-2u_1u_3)(\dot{u}_2+u_1u_2\cot_\theta-u_1u_3)\Big)$$

EXAMPLE: ROLLING DISK WITHOUT SLIP

Generalized forces: $Q_k=\sum_{i=1}^{N}\vec{\mathbf{F}}_i\cdot\vec{\boldsymbol{\gamma}}_{ik}+\vec{\mathbf{M}}_i\cdot\vec{\boldsymbol{\beta}}_{ik}\quad k=1,\ldots,n-m$

$$\left.\begin{array}{l}\vec{\mathbf{F}}=-mg\hat{\mathbf{K}}=-mg(S_\theta\hat{\mathbf{j}}_0+C_\theta\hat{\mathbf{k}}_0),\quad\vec{\mathbf{M}}=\mathbf{0}\\ \vec{\mathbf{v}}_G:\quad\vec{\boldsymbol{\gamma}}_1=R\hat{\mathbf{k}}_0,\ \vec{\boldsymbol{\gamma}}_2=-R\cot_\theta\hat{\mathbf{i}}_0,\ \vec{\boldsymbol{\gamma}}_3=\mathbf{0}\end{array}\right\}\Rightarrow\begin{cases}Q_1=-mgRC_\theta\\ Q_2=0\\ Q_3=0\end{cases}$$

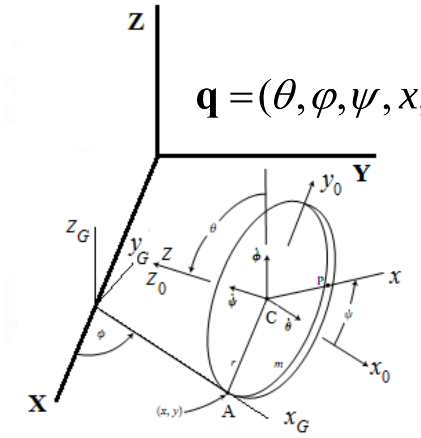
EOM: $\frac{\partial S}{\partial\dot{u}_k}=Q_k$

$$\frac{\partial S}{\partial\dot{u}_1}=Q_1\ \Rightarrow\ 5\dot{u}_1-u_2^2\cot_\theta+6u_2u_3=-\frac{4g}{R}C_\theta$$

$$\frac{\partial S}{\partial\dot{u}_2}=Q_2\ \Rightarrow\ \dot{u}_2+u_1u_2\cot_\theta-2u_1u_3=0$$

$$\frac{\partial S}{\partial\dot{u}_3}=Q_3\ \Rightarrow\ 3\dot{u}_3-2u_1u_2=0$$

EXAMPLE: ROLLING DISK WITHOUT SLIP



$$\mathbf{q} = (\theta, \varphi, \psi, x, y)^T$$

$$\vec{\boldsymbol{\omega}}_B = \dot{\theta}\hat{\mathbf{i}}_0 + \dot{\varphi}S_\theta \hat{\mathbf{j}}_0 + (\dot{\varphi}C_\theta + \dot{\psi})\hat{\mathbf{k}}_0$$

$$u_1 = \omega_{B_{x0}} = \dot{\theta}$$

$$u_2 = \omega_{B_{y0}} = \dot{\varphi}S_\theta$$

$$u_3 = \omega_{B_{z0}} = \dot{\varphi}C_\theta + \dot{\psi}$$

$$\begin{cases} \dot{u}_1 = \frac{1}{5}\left(u_2^2 \cot_\theta - 6u_2 u_3 - \frac{4g}{R} C_\theta \right) \\ \dot{u}_2 = -u_1 u_2 \cot_\theta + 2u_1 u_3 \\ \dot{u}_3 = \frac{2}{3} u_1 u_2 \end{cases}$$

Dynamic equations

Kinematic equations

$$\begin{cases} \dot{q}_1 = u_1 \\ \dot{q}_2 = u_2 / S_\theta \\ \dot{q}_3 = -(\cot\theta)u_2 + u_3 \\ \dot{q}_4 = R\cot_\theta C_\varphi u_2 - RC_\varphi u_3 \\ \dot{q}_5 = R\cot_\theta S_\varphi u_2 - RS_\varphi u_3 \end{cases}$$

SIMILARITY BETWEEN KANE'S AND GIBBS APPELL EQUATIONS

Do you see any difference?

$$Q_k^* = Q_k$$

$$Q_k^* = \frac{\partial S}{\partial \dot{u}_k} = \sum_{i=1}^{N_B} m_i \dot{\vec{\mathbf{v}}}_i \bullet \vec{\boldsymbol{\gamma}}'_{ik} + \sum_{i=1}^{N_B} \left(\dot{\vec{\boldsymbol{\omega}}}_i \bullet \vec{\mathbf{I}}_{ci} + \vec{\boldsymbol{\omega}}_i \times (\vec{\mathbf{I}}_{ci} \bullet \vec{\boldsymbol{\omega}}_i) \right) \bullet \vec{\boldsymbol{\beta}}'_{ik}$$

$$= \sum_{i=1}^{N_F} \vec{\mathbf{F}}_i^* \bullet \vec{\mathbf{v}}_i^k + \sum_{i=1}^{N_M} \vec{\mathbf{M}}_i^* \bullet \vec{\boldsymbol{\omega}}_i^k$$

$$Q_k^* = Q_k \qquad Q_k = \sum_{i=1}^{N_F} \vec{\mathbf{F}}_i \bullet \vec{\mathbf{v}}_i^k + \sum_{i=1}^{N_M} \vec{\mathbf{M}}_i \bullet \vec{\boldsymbol{\omega}}_i^k$$

$$Q_k^* = -\sum_{i=1}^{N_B} m_i \vec{\boldsymbol{a}}_{ci} \bullet \vec{\mathbf{v}}_i^k + \frac{d}{dt} \vec{\mathbf{H}}_{ci} \bullet \vec{\boldsymbol{\omega}}_i^k$$

$$\frac{d}{dt} \vec{\mathbf{H}}_{ci} = \vec{\mathbf{I}}_{ci} \bullet \dot{\vec{\boldsymbol{\omega}}}_i + \vec{\boldsymbol{\omega}}_i \times (\vec{\mathbf{I}}_{ci} \bullet \vec{\boldsymbol{\omega}}_i)$$